SOLUTIONS TO PROBLEMS

CONTAINED IN

A TREATISE ON

PLANE COORDINATE GEOMETRY.

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PLANE COORDINATE GEOMETRY

BY

Isaac TODHUNTER, M.A., F.R.S.

EDITED BY

C. W. BOURNE, M.A.
HEAD MASTER OF INVERNESS COLLEGE.

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PREFACE.

THE greater part of this collection of solutions was made about fifteen years ago for the benefit of my pupils at Marlborough College, and has been tested by tolerably frequent use since that time.

I have thought that it may be serviceable to give a short introduction, detailing certain of the more important facts in the Theory of Equations, so far as they bear on the subject.

I have endeavoured to render the solutions intelligible to those who are working the subject through for the first time; consequently the earlier chapters are treated in fuller detail than the later ones, and I have at times given alternative solutions, where such seemed likely to be instructive.

For the same reason geometrical methods have been occasionally employed in preference to analytical ones, in order that the student may not become a mere "manipulator of equations".

I have endeavoured to secure as much accuracy as possible by re-working each example as I copied it out for the press, and by again re-working each from the proof-sheets. I shall be very grateful for any hints or corrections.

C. W. BOURNE.

INVERNESS COLLEGE,
July, 1887.

PLANE CO-ORDINATE GEOMETRY.

INTRODUCTION.

As success in solving problems in Analytical Geometry is largely dependent upon skill in handling equations, a short summary is here given of the principal facts in the Theory of Quadratic Equations, so far as they are applicable.

The typical quadratic equation may be taken as $ax^2+bx+c=0$, or $x^2+\frac{bx}{a}+\frac{c}{a}=0$.

If this equation is solved, the roots are found to be

$$-\frac{b}{2a}+\frac{\sqrt{(b^2-4ac)}}{2a} \text{ and } -\frac{b}{2a}-\frac{\sqrt{(b^2-4ac)}}{2a}.$$

Hence we have the following results :

I. If $b^2=4ac$, the two roots are equal, each being equal to $-\frac{b}{2a}$. In other words the quantity $x^2+\frac{bx}{a}+\frac{c}{a}$ is in this case a perfect square.

II. The sum of the roots is $-\frac{b}{a}$; their product is $\frac{c}{a}$.

III. If b^2-4ac is a perfect square, the roots are rational.

IV. The difference of the roots is $\frac{\sqrt{(b^2-4ac)}}{a}$.

V. The preceding fact may be presented in another shape, which is often useful. Let α, β be the roots, then the difference of the roots is $\alpha-\beta$. Also it is evident that

$$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta,$$

or (difference of roots)^2 = (sum of roots)^2 − four times their product.

VI. If $b^2<4ac$, the roots are impossible. If b^2 is not $<4ac$, the roots are real.

VII. If $a=0$, one root becomes infinite.

VIII. If the equations $y^2+mx+n=0$ and $y^2+ax+b=0$ are *identical*,—that is to say, if they both express the same locus referred to the same axes,—then we must have $m=a$ and $n=b$. The sign $\equiv$ is frequently used to express the identity of two quantities.

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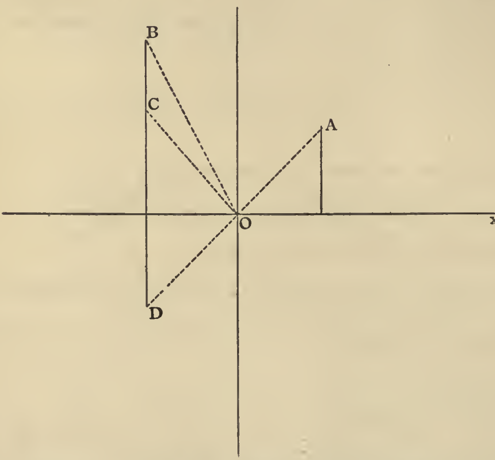
CONIC SECTIONS.

CHAPTER I.

1. If we make use of the equations of Art. 8 to find r and θ, we shall have an ambiguity in the values, which must be cleared up by reference to the



particular quadrant in which the point is shewn to be by its rectangular co-ordinates.

(1) If $x=1$, $y=1$, then $r^2=2$, $\therefore r=\pm\sqrt{2}$. $\tan\theta=1$, $\therefore \theta=45^0$ or 225^0, or -135^0, or -315^0.

It is evident by the rectangular co-ordinates that the point in question is A, and that it is reached by revolving through 45^0 and then measuring $\sqrt{2}$ along the revolving line; hence its polar co-ordinates are $(\sqrt{2}, 45^0)$.

It might also be reached by revolving through an angle 225^0, so as to reach the position OD, and then measuring our distance, *not* along the revolving line OD, *but* along its part *OA produced backwards;* hence the point can also be defined by the co-ordinates $(-\sqrt{2}, 225^0)$.

Since the position OD can be reached by revolving through a negative angle -135^0, the point can also be defined by the co-ordinates $(-\sqrt{2}, -135^0)$.

It could also be defined by the co-ordinates $(\sqrt{2}, -315^0)$.

(2) Here $r^2=1+4, \therefore r=\pm\sqrt{5}$.

$\tan\theta=-2$. But the angle whose tangent is $+2$ is very nearly $63\frac{1}{2}^0$; hence θ is approximately equal to $-63\frac{1}{2}^0$, or $116\frac{1}{2}^0$, or $296\frac{1}{2}^0$, or $-243\frac{1}{2}^0$.

The point is the point B, and can be defined by the co-ordinates

$$(\sqrt{5}, 116\tfrac{1}{2}^0), \text{ or } (\sqrt{5}, -243\tfrac{1}{2}^0), \text{ or } (-\sqrt{5}, -63\tfrac{1}{2}^0), \text{ or } (-\sqrt{5}, 296\tfrac{1}{2}^0).$$

(3) Here $r^2=1+1, \therefore r=\pm\sqrt{2}$.

$\tan\theta=-1$; $\therefore \theta=135^0$, or 315^0, or -45^0, or -225^0.

The point is the point C, and can be defined by the co-ordinates

$$(\sqrt{2}, 135^0), \text{ or } (\sqrt{2}, -225^0), \text{ or } (-\sqrt{2}, -45^0), \text{ or } (-\sqrt{2}, 315^0).$$

(4) Here $r^2=1+1, \therefore r=\pm\sqrt{2}$.

$\tan\theta=1$; $\therefore \theta=45^0$, or 225^0, or -135^0, or -315^0.

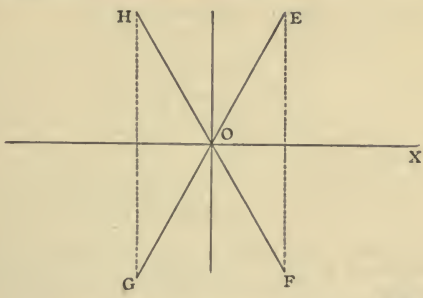
The point is the point D, and can be defined by the co-ordinates

$$(\sqrt{2}, 225^0), \text{ or } (\sqrt{2}, -135^0), \text{ or } (-\sqrt{2}, 45^0), \text{ or } (-\sqrt{2}, -315^0).$$

2. (1) Using the formulæ of Art. 8, we have

$$x=3\cos 60^0=\tfrac{3}{2},\ y=3\sin 60^0=\frac{3\sqrt{3}}{2}.$$

The point is the point E in the adjoining figure.



(2) Here $x=3\cos(-60^0)=\frac{3}{2}$; $y=3\sin(-60^0)=-\dfrac{3\sqrt{3}}{2}$.

The point is the point F.

(3) Here $x=-3\cos 60^0=-\frac{3}{2}$; $y=-3\sin 60^0=-\dfrac{3\sqrt{3}}{2}$.

The point is the point G.

(4) Here $x=-3\cos(-60^0)=-\frac{3}{2}$; $y=-3\sin(-60^0)=\dfrac{3\sqrt{3}}{2}$.

The point is the point H.

3. By Art. 9

$$PQ^2=(3+1)^2+(7-4)^2=25,$$
$$\therefore\ PQ=5.$$

4. By Art. 11 the area required

$$=\pm\tfrac{1}{2}\{-1-2-2+1+1+1\}=\pm\tfrac{1}{2}(-2)=\pm 1.$$

The area is therefore 1.

5. Let abscissa of A be h; its ordinate is 0.

Let ordinate of B be k; its abscissa is 0.

$\therefore$ co-ordinates of middle point of AB are $(\tfrac{1}{2}h, \tfrac{1}{2}k)$.

Let this point be D; then

$$OD^2=(\tfrac{1}{2}h)^2+(\tfrac{1}{2}k)^2=\frac{h^2+k^2}{4}.$$
$$\therefore\ OD=\tfrac{1}{2}\sqrt{h^2+k^2}=\tfrac{1}{2}AB.$$

6. Let $x_1=r_1\cos\theta_1$, $x_2=r_2\cos\theta_2$, and so on; then the area of triangle

$$=\pm\tfrac{1}{2}\{r_1r_2\sin\theta_1\cos\theta_2-r_1r_2\sin\theta_2\cos\theta_1+r_2r_3\sin\theta_2\cos\theta_3$$
$$-r_2r_3\sin\theta_3\cos\theta_2+r_1r_3\sin\theta_3\cos\theta_1-r_1r_3\sin\theta_1\cos\theta_3\}$$
$$=\pm\tfrac{1}{2}\{r_1r_2\sin\overline{\theta_1-\theta_2}+r_2r_3\sin\overline{\theta_2-\theta_3}+r_1r_3\sin\overline{\theta_3-\theta_1}\}.$$

This may also be got directly thus (figure to Art. 11):

$$OA=r_1,\quad OB=r_2,\quad \text{angle } AOB=\theta_1-\theta_2;$$

$\therefore$ area of $AOB=\tfrac{1}{2}r_1r_2\sin\overline{\theta_1-\theta_2}$.

Similarly area of $BOC=\tfrac{1}{2}r_2r_3\sin\overline{\theta_2-\theta_3}$;

and area of $AOC=\tfrac{1}{2}r_1r_3\sin\overline{\theta_1-\theta_3}$.

But area of $ABC=AOB+BOC-AOC$

$$=\tfrac{1}{2}\{r_1r_2\sin\overline{\theta_1-\theta_2}+r_2r_3\sin\overline{\theta_2-\theta_3}-r_1r_3\sin\overline{\theta_1-\theta_3}\}$$
$$=\tfrac{1}{2}\{r_1r_2\sin\overline{\theta_1-\theta_2}+r_2r_3\sin\overline{\theta_2-\theta_3}+r_1r_3\sin\overline{\theta_3-\theta_1}\}.$$

7. If (x_1, y_1) and (x_2, y_2) be the co-ordinates of A and B, we can obtain our result immediately from Art. 11 by making $x_3=0$, $y_3=0$, since our third point is the origin.

The area then $=\tfrac{1}{2}\{x_2y_1-x_1y_2\}$.

If the polar co-ordinates of A and B are (r_1, θ_1) and (r_2, θ_2) we have area of

$$AOB=\tfrac{1}{2}AO\,.\,OB\,.\,\sin AOB=\tfrac{1}{2}r_1r_2\sin(\theta_1-\theta_2).$$

8. The co-ordinates of D are evidently

$$\frac{x_1+x_2}{2},\quad \frac{y_1+y_2}{2};$$

then in the result of Art. 10 let us replace x_1 and y_1 by the co-ordinates of C, namely x_3 and y_3; and let us replace x_2 and y_2 by the co-ordinates of D,

namely $$\frac{x_1+x_2}{2} \text{ and } \frac{y_1+y_2}{2};$$

also in this case $$n_1=2n_2.$$

Hence $$x=\frac{x_1+x_2+x_3}{3}, \text{ and } y=\frac{y_1+y_2+y_3}{3}.$$

9. The area of GAB is, by Art. 11,

$$\tfrac{1}{2}\left\{x_2y_1-x_1y_2+\frac{x_1+x_2+x_3}{3}\cdot y_2-\frac{y_1+y_2+y_3}{3}\cdot x_2+\frac{y_1+y_2+y_3}{3}\cdot x_1 - \frac{x_1+x_2+x_3}{3}\cdot y_1\right\}$$

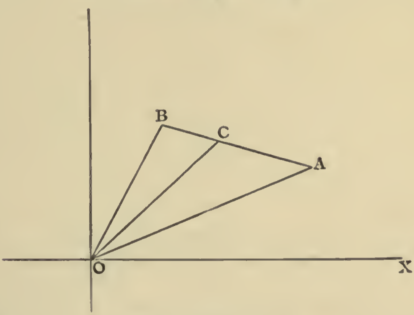
$$=\tfrac{1}{6}\{3x_2y_1-3x_1y_2+x_1y_2+x_2y_2+x_3y_2-x_2y_1-x_2y_2-x_2y_3+x_1y_1+x_1y_2+x_1y_3 - x_1y_1-x_2y_1-x_3y_1\}$$

$$=\tfrac{1}{6}\{x_2y_1-x_1y_2+x_3y_2-x_2y_3+x_1y_3-x_3y_1\}=\tfrac{1}{3} \text{ of triangle } ABC.$$

Similarly for the other triangles.

10. Since angle $$COA=\tfrac{1}{2}BOA=\frac{\theta_2-\theta_1}{2},$$

$$\therefore\ COX=\theta_1+\frac{\theta_2-\theta_1}{2}=\frac{\theta_2+\theta_1}{2}.$$



Also area of $$COA+BOC=BOA,$$

$$\therefore\ \tfrac{1}{2}rr_1\sin\frac{\theta_2-\theta_1}{2}+\tfrac{1}{2}rr_2\,.\,\sin\frac{\theta_2-\theta_1}{2}=\tfrac{1}{2}r_1r_2\,.\,\sin\overline{\theta_2-\theta_1}$$

$$=\tfrac{1}{2}r_1r_2\times 2\sin\frac{\theta_2-\theta_1}{2}\,.\,\cos\frac{\theta_2-\theta_1}{2};$$

divide by $\tfrac{1}{2}\sin\frac{\theta_2-\theta_1}{2}$, and we get

$$rr_1+rr_2=2r_1r_2\cos\frac{\theta_2-\theta_1}{2},$$

whence the required result is at once obtained.

11. Since co-ordinates of C and D are (x_3, y_3), and $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ respectively, we have

$$CD^2 = \left(\frac{x_1+x_2-2x_3}{2}\right)^2 + \left(\frac{y_1+y_2-2y_3}{2}\right)^2.$$

Similarly
$$AD^2 = \left(\frac{x_1+x_2-2x_1}{2}\right)^2 + \left(\frac{y_1+y_2-2y_1}{2}\right)^2$$
$$= \left(\frac{x_2-x_1}{2}\right)^2 + \left(\frac{y_2-y_1}{2}\right)^2.$$

Hence it is easily seen that

$$2AD^2 + 2CD^2 = (x_1^2 + x_2^2 + 2x_3^2 - 2x_1x_3 - 2x_2x_3) + (y_1^2 + y_2^2 + 2y_3^2 - 2y_1y_3 - 2y_2y_3),$$

and the expression for $AC^2 + BC^2$ will be found to be the same.

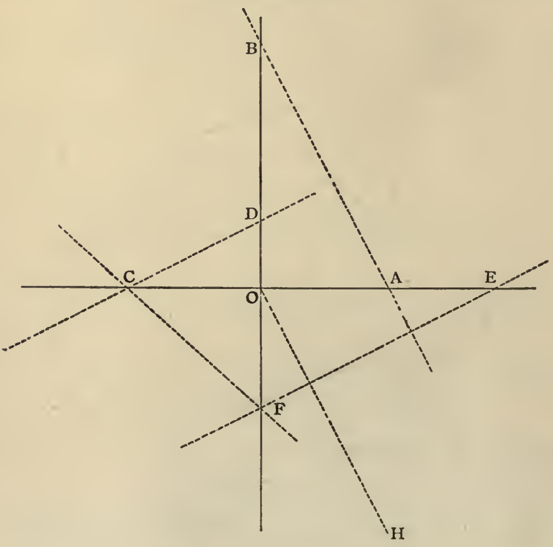
12. We have

$$GA^2 = \left(\frac{x_1+x_2+x_3}{3} - x_1\right)^2 + \left(\frac{y_1+y_2+y_3}{3} - y_3\right)^2,$$

$$\therefore\ 3GA^2 = \frac{x_2^2 + x_3^2 + 4x_1^2 + 2x_2x_3 - 4x_1x_2 - 4x_1x_3}{3} + \text{similar function of } y.$$

Similarly for $3GB^2$, and $3GC^2$; hence the required result easily follows.

CHAPTER II.



1. The straight line cuts the axis of x at the point whose abscissa is 2, as we find by making $y=0$.

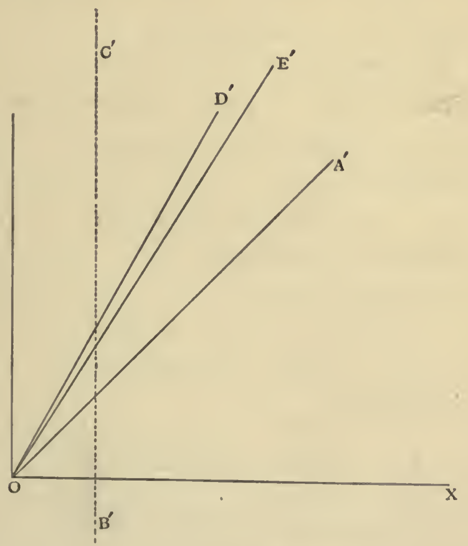
Similarly it cuts the axis of y at the point whose ordinate is 4, as shewn by making $x=0$. Hence it is the line AB.

2. The line cuts the axis of x at the point $(-2, 0)$, and axis of y at point $(0, 1)$; hence it is the line CD.

3. The line cuts the axis at the points $(-2, 0)$ and $(0, -2)$; hence it is the line CF.

4. The line cuts the axes at the points $(4, 0)$ and $(0, -2)$; hence it is the line EF.

5. When $x=0$, $y=0$; hence the line goes through the origin. Also when $x=2$, $y=-4$; and these are the co-ordinates of H. Hence OH is the line.



6. Solving this equation we get $\theta-\frac{\pi}{4}=2n\pi$, where n may have any value.

$\therefore\ \theta=\frac{\pi}{4}+2n\pi.$

But as $2n\pi$ only means a certain number of *complete* revolutions, the solution is merely equivalent to $\theta=\frac{\pi}{4}$.

If the angle $A'OX$ be 45^0 it is evident that for *every* point on OA' we have $\theta=\frac{\pi}{4}$; hence OA' is the required line.

7. By Art. 15 the straight line is parallel with the axis of y, and is evidently the line $B'C'$.

8. If $D'OX=60^0$, then OD' is evidently the line.

9. This equation evidently represents the axis of x.

10. This equation means that θ is equal to the unit of circular measure, which is an angle of about $57\frac{1}{3}^0$; hence if $E'OX$ is an angle of this size, the line OE' is the required line.

CHAPTER III.

1. (1) The equation is

$$y-1=\frac{-1-1}{1-0}(x-0) \text{ or } y+2x-1=0. \quad \text{(See Art. 35.)}$$

(2) By Art. 15 the equation is evidently $x=2$.

(3) The equation is

$$y-1=\frac{-2-1}{-2-1}(x-1), \text{ or } y=x.$$

(4) By Art. 15 the equation is $x=0$.

2. By Art. 45 the equations are

$$y-4=\frac{2+1}{1-2}(x-4), \text{ and } y-4=\frac{2-1}{1+2}(x-4),$$

or

$$y-4=-3(x-4), \text{ and } y-4=\tfrac{1}{3}(x-4).$$

3. The equations are

$$y-1=\frac{-1+\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}x, \text{ and } y-1=\frac{-1-\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}x,$$

or

$$y-1=(\sqrt{3}-2)x, \text{ and } y-1=-(\sqrt{3}+2)x.$$

4. If a line passes through the origin, c is zero, and therefore the line is of the form $y=mx$. Here the lines are to make angles of 45^0 and -45^0 with the axis of y (since the given line is parallel to the axis of y); therefore they will make angles of 45^0 and -45^0 with the axis of x; that is to say, the value of m is ± 1; hence the required lines are

$$y=x \text{ and } y=-x.$$

5. The equations are

$$y=\frac{-\frac{1}{\sqrt{3}}+\sqrt{3}}{1+1}x, \text{ and } y=\frac{-\frac{1}{\sqrt{3}}-\sqrt{3}}{1-1}x;$$

the first of these becomes $y=\frac{1}{\sqrt{3}}x$, and the second takes the form $y=\infty\times x$, which is unintelligible; if however we write it $x=\frac{y}{\infty}=0$, we see that it is the axis of y.

6. If θ be the angle required, then by Art. 41 we have

$$\tan\theta=\frac{1+1}{1-1}=\infty; \quad \therefore \theta=90^0.$$

Also, solving the equations simultaneously, we get $x=-\frac{1}{2}$, $y=\frac{3}{2}$, and therefore the point of intersection is $(-\frac{1}{2}, \frac{3}{2})$.

7. Here $$\tan\theta=\frac{\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}}=\sqrt{3}; \quad \therefore \theta=60^0.$$

8. Here $$\tan\theta=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=1; \quad \therefore \theta=45^0.$$

9. Let $(a, 0)$ be the co-ordinates of the given point; then by Art. 32, since m is to be $+1$ or -1, we have the equations

$$y=x-a, \text{ and } y=-(x-a).$$

10. By Art. 44 the equation is evidently $y=x$.

11. By Art. 47 the perpendicular $=\pm\frac{1-2-3}{\sqrt{1^2+1^2}}=2\sqrt{2}$.

12. By Art. 49 the perpendicular $=\frac{\frac{a}{a}+\frac{b}{b}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=\frac{ab}{\sqrt{a^2+b^2}}$.

13. Solving simultaneously, we get

$$x=\frac{ab}{a+b}, \quad y=\frac{ab}{a+b}.$$

14. Using the result of the last example, our equation is

$$y-b=\frac{\frac{ab}{a+b}-b}{\frac{ab}{a+b}-a}(x-a),$$

or $$\frac{x}{a^2}-\frac{y}{b^2}=\frac{1}{a}-\frac{1}{b}.$$

15. (1) This equation can only be satisfied by $x=0$, and $y=0$, *simultaneously*; but the only point for which *both* co-ordinates are 0 is the origin; hence the equation represents the origin.

(2) If $x^2-y^2=0$, then $(x+y)(x-y)=0$; hence *either* $x+y=0$, *or else* $x-y=0$. The first of these two equations is the line $y=-x$, and the second is the line $y=x$.

(3) If $x^2+xy=0$, then $x(x+y)=0$; hence we have two straight lines

$$x=0, \quad \text{and} \quad x+y=0.$$

(4) If $xy=0$, then we have two suppositions,—either $x=0$ and y may have any value (which denotes the axis of y); or $y=0$ and x can have any value (which denotes the axis of x). Hence the equation represents the two axes.

(5) The values of x and y are obviously impossible. There is therefore no locus.

(6) If $x(y-a)=0$, then the locus consists of the two lines $x=0$, and $y-a=0$.

16. (1) The locus consists of the two lines $x-a=0$, and $y-b=0$.

(2) Here $x-a=0$, and $y-b=0$, *simultaneously*; hence the locus is the *point* (a, b).

(3) Here $x-y+a=0$, and $x+y-a=0$, *simultaneously*; hence the locus is the point of intersection of these two lines; this is the point $(0, a)$.

17. Resolving into factors, the equation can be written

$$(y-3x)(y-x)=0.$$

Hence it represents the two straight lines $y=3x$, and $y=x$.

18. Resolving into factors, the equation becomes

$$(3y+x-9)(y-3x+3)=0,$$

which gives the two straight lines

$$y=-\tfrac{1}{3}x+3, \quad \text{and} \quad y=3x-3.$$

By Art. 42 these are evidently perpendicular to each other.

19. Let A be the point of intersection of $y=2x$ with $y=5$; therefore co-ordinates of A are $(\frac{5}{2}, 5)$.

Similarly if B be the intersection of $x=4$ with $y=5$, the co-ordinates of B are $(4, 5)$.

If C be intersection of $y=x$ with $x=4$, co-ordinates of C are $(4, 4)$.

Hence equation to OB will be found to be $4y=5x$, and equation to AC will be $2x+3y-20=0$. (See Art. 35.)

20. Let us take a as the length of a side; then remembering that each angle of a regular hexagon is 120^0, we obtain the following co-ordinates:

Of A they are $(0, 0)$; of B they are $(a, 0)$; of C they are $\left(\frac{3a}{2}, \frac{a\sqrt{3}}{2}\right)$; of D they are $(a, a\sqrt{3})$; of E they are $(0, a\sqrt{3})$; of F they are $\left(-\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$.

Hence the required equations can be at once obtained.

21. If the angular points of the triangle are (x_1, y_1), (x_2, y_2), (x_3, y_3), then the co-ordinates of the middle points of the sides are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right), \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right), \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

The equation to the straight line joining the last two points is

$$y-\frac{y_1+y_2}{2}=\frac{\frac{y_1+y_3}{2}-\frac{y_1+y_2}{2}}{\frac{x_1+x_3}{2}-\frac{x_1+x_2}{2}}\left(x-\frac{x_1+x_2}{2}\right),$$

or

$$y-\frac{y_1+y_2}{2}=\frac{y_3-y_2}{x_3-x_2}\left(x-\frac{x_1+x_2}{2}\right).$$

Similarly the other two lines may be found.

22. By Art. 56 the tangent required $=\dfrac{\left(m+\frac{1}{m}\right)\sin\omega}{1+\left(m-\frac{1}{m}\right)\cos\omega-1}$,

which reduces to $\dfrac{m^2+1}{m^2-1}\cdot\tan\omega$.

23. Taking the angle between the axes as ω, we have the tangent of included angle $=\dfrac{(1+1)\sin\omega}{1+(1-1)\cos\omega-1}$, which $=\infty$ whatever ω is.

24. Let $ABCD$ be the parallelogram, AB the axis of x, AD the axis of y; let the lengths of AB, AD be a, b, and their included angle be ω.

Then, by Art. 35, the equation to AC is $y=\frac{b}{a}\cdot x$; and equation to BD is (Art. 17) $\frac{y}{b}+\frac{x}{a}=1$; therefore tangent of included angle

$$=\frac{\left(\frac{b}{a}+\frac{b}{a}\right)\sin\omega}{1+\left(\frac{b}{a}-\frac{b}{a}\right)\cos\omega-\frac{b^2}{a^2}}=\frac{2ab\sin\omega}{a^2-b^2}.$$

25. See figure of Art. 75. The co-ordinates of A are $(a, 0)$; of B they are $(0, 0)$; of C they are $(0, c)$; of D they are (h, k).

$\therefore$ equation to CA is (by Art. 25) $\frac{x}{a}+\frac{y}{c}=1$;

equation to BD is $\qquad y=\frac{k}{h}x$;

equation to AD is $\qquad y=\frac{k}{h-a}(x-a)$;

equation to CD is $\qquad y-c=\frac{k-c}{h}.x$.

26. By Art. 10 the co-ordinates of middle point of AC are $\left(\frac{a}{2}, \frac{c}{2}\right)$, of middle of BD they are $\left(\frac{h}{2}, \frac{k}{2}\right)$; and the equation to the line joining these two points is
$$y-\frac{c}{2}=\frac{k-c}{h-a}\left(x-\frac{a}{2}\right).$$

27. F is the intersection of $x=0$ with $y=\frac{k}{h-a}(x-a)$; hence, solving these equations simultaneously, we get the co-ordinates of F as $\left(0, \frac{ak}{a-h}\right)$.

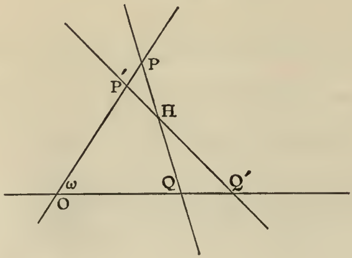
Similarly E is the intersection of $y=0$ with $y-c=\frac{k-c}{h}.x$; hence its co-ordinates are $\left(\frac{hc}{c-k}, 0\right)$.

Hence the co-ordinates of the middle point of EF are
$$\left(\frac{hc}{2c-2k}, \frac{ak}{2a-2h}\right).$$

It will be found that these co-ordinates satisfy the equation obtained in the preceding example.

28. Let PQ be the straight line $\frac{x}{a}+\frac{y}{b}=1$, so that $OP=b$, $OQ=a$.

Similarly $\qquad OP'=b'$, $OQ'=a'$.



Now since the area $POQ = P'OQ'$ we have

$$\tfrac{1}{2}ab \sin \omega = \tfrac{1}{2}a'b' \sin \omega, \text{ or } ab = a'b'.$$

Also, solving the two given equations simultaneously we get

$$x' = \frac{(b'-b)\,aa'}{ab'-ba'}, \text{ and } y = \frac{(a-a')\,bb'}{ab'-a'b};$$

hence we have
$$\frac{y'}{x'} = \frac{abb'-a'bb'}{aa'b'-a'ab},$$

and dividing every term by one of the two equal quantities ab and $a'b'$, we get

$$\frac{y'}{x'} = \frac{b'-b}{a-a'}.$$

29. Let the required point be $(h, 0)$; its distance from the line is

$$\frac{\frac{h}{a}-1}{\pm\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}},$$

and if this is equal to a we get

$$h = a \pm \frac{1}{b}\sqrt{a^2+b^2}.$$

30. Let (h, k) be the required point; then since it is on the given line we have
$$\frac{h}{a}+\frac{k}{b}=1.$$

Also
$$(h-\alpha)^2+(k-\beta)^2=c^2.$$

Solving these two equations simultaneously we get

$$h^2(a^2+b^2)-2ha\{a\alpha+b^2-b\beta\}+a^2\{\alpha^2+(b-\beta)^2-c^2\}=0;$$

as this is a quadratic it will in general have *two* solutions, so that there will be *two* points to satisfy the conditions.

If however the points coincide, the roots of the above quadratic must be equal; the conditions for this are (by Introd. § I.)

$$4(a^2+b^2)\,.\,a^2\,.\,\{\alpha^2+(b-\beta)^2-c^2\}=4a^2\{a\alpha+b^2-b\beta\}^2,$$

which reduces to the required form.

31. Let the two lines be

$$y-m_1x=0, \text{ and } y-m_2x=0.$$

Hence $(y-m_1x)(y-m_2x)$ is to be identical with

$$y^2+\frac{B}{A}xy+\frac{C}{A}x^2.$$

Hence (by Introd. § VIII.)

$$m_1+m_2 \equiv -\frac{B}{A}, \text{ and } m_1m_2 \equiv \frac{C}{A}.$$

But if θ is the angle between the given lines, we have

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\sqrt{B^2 - 4AC}}{A + C};$$

(see Introd. § v.).

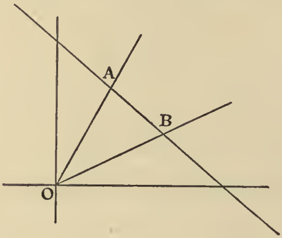
[*Note.* If the lines are at right angles, $A + C = 0$, or $A = -C$.]

32. By solving the equations simultaneously in pairs we get the points of intersection to be (3, 1), (1, 2), (2, 3).

Substituting in the formula of Art. 11, we get the area $= \frac{3}{2}$.

33. Solving the equations simultaneously in pairs we get co-ordinates of

$$A = \left(\frac{c \,.\, \cos\alpha \,.\, \cos\gamma}{\sin(\alpha - \gamma)}, \frac{c \,.\, \sin\alpha \,.\, \cos\gamma}{\sin(\alpha - \gamma)}\right),$$



from which we get

$$OA = \frac{c \,.\, \cos\gamma}{\sin(\alpha - \gamma)}; \text{ similarly } OB = \frac{c \,.\, \cos\gamma}{\sin(\beta - \gamma)}.$$

But area of $AOB = \frac{1}{2} OA \,.\, OB \sin AOB$

$$= \frac{1}{2} \cdot \frac{c^2 \cos^2\gamma \sin(\alpha - \beta)}{\sin(\alpha - \gamma) \,.\, \sin(\beta - \gamma)}.$$

34. The distance between two parallel straight lines is evidently the difference between the perpendiculars from the origin upon them; but if the two lines be $y = mx + c_1$ and $y = mx + c_2$ the perpendiculars from the origin will (by Art. 47) be $\frac{c_1}{\sqrt{1 + m^2}}$ and $\frac{c_2}{\sqrt{1 + m^2}}$.

Hence the distance required is $\frac{c_1 - c_2}{\sqrt{1 + m^2}}$.

35. By Art. 8 the equations may be written

$$4x + 3y = a, \text{ and } 3x - 4y = b,$$

or

$$y = -\tfrac{4}{3}x + \frac{a}{3}, \text{ and } y = \tfrac{3}{4}x - \frac{b}{4}.$$

Hence, by Art. 42, the lines are at right angles.

36. $F(\theta)=0$ is an equation which has as many solutions as it has dimensions in θ, and each solution gives a straight line through the origin (see Example 6 of Chap. II.).

If $\sin 3\theta=0$, the solutions are $\theta=0^0, 60^0, 120^0$, representing three straight lines through the origin, the first being the initial line, and the other two making angles of 60^0 and 120^0 with the initial line.

37. By Art. 56 the tangent of the inclination of $Ax+By+C=0$ to the axis of x, that is to say to the straight line $y=0$, is

$$\frac{-\frac{A}{B}\sin\omega}{1-\frac{A}{B}\cos\omega} \text{ or } \frac{A\sin\omega}{A\cos\omega-B}.$$

Similarly for the other straight line we shall get $-\dfrac{A'\sin\omega}{A'\cos\omega-B'}$, because the angle is to be made on the *negative* side of the axis.

$$\therefore \frac{A\sin\omega}{A\cos\omega-B}=-\frac{A'\sin\omega}{A'\cos\omega-B'},$$

which becomes
$$\frac{B}{A}+\frac{B'}{A'}=2\cos\omega.$$

38. If the lines go through the origin we must have $C=0$, and $C'=0$, so that the equations are $Ax+By=0$, and $A'x+B'y=0$; which may be combined in the one equation

$$\left(x+\frac{B}{A}y\right)\left(x+\frac{B'}{A'}y\right)=0.$$

But each of the tangents in the preceding example is to become $\tan 45^0$ or 1, hence $\dfrac{A\sin\omega}{A\cos\omega-B}=1$, from which we get $\dfrac{B}{A}=\cos\omega-\sin\omega$. Similarly we have
$$\frac{B'}{A'}=\cos\omega+\sin\omega.$$

Hence our equation becomes

$$(x+y\,.\,\overline{\cos\omega-\sin\omega})\,(x+y\,.\,\overline{\cos\omega+\sin\omega})=0,$$

or
$$x^2+2xy\cos\omega+y^2\cos 2\omega=0.$$

39. The equations to the two lines are

$$y-b=\tan\theta\,(x-a) \text{ and } y-b'=\tan\theta\,(x-a').$$

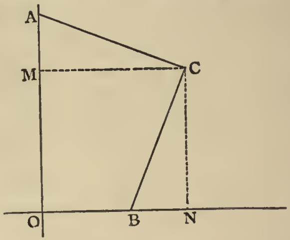
Using the method of Example 34, the distance between them is found to be
$$(b'-b)\cos\theta-(a'-a)\sin\theta.$$

Secondly, let the four sides of the rectangle be $y-b_1=\tan\theta\,(x-a_1)$, and $y-b_2=-\cot\theta\,(x-a_2)$, and $y-b_3=\tan\theta\,(x-a_3)$, and $y-b_4=-\cot\theta\,(x-a_4)$.

Then the length of the rectangle will be (by preceding part)

$(b_3-b_1)\cos\theta-(a_3-a_1)\sin\theta$, and its breadth $(b_2-b_4)\sin\theta-(a_2-a_4)\cos\theta$; if the product of these two expressions be equated to the given area we have an equation to find θ.

40. Let the two fixed straight lines be taken as axes; let a be the length of the side of a square.



Now since ACB and MCN are both right angles, we get $ACM = BCN$.

Hence the triangles ACM and BCN are easily seen to be equal in all respects. $\therefore\ CM = CN$.

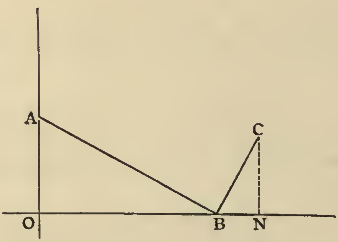
Hence the equation to OC is $y = x$.

Similarly if D be the other corner of square, equation to OD is $y = -x$; and these two lines are at right angles.

41. Let the line along which B moves be taken as axis of x, and a line through A perpendicular to this as the axis of y. Let O be the origin.

Let $OA = a$, $OAB = \theta$; $\therefore\ CBN = \theta$. Let the ratio $AB : BC$ be equal to m.

Now $$AB = a\sec\theta,\ \therefore\ BC = \frac{a\sec\theta}{m}.$$



Let (x, y) be co-ordinates of C.

$$\therefore\ x = OB + BN = a\tan\theta + \frac{a\sec\theta}{m}\times\cos\theta$$

$$= a\tan\theta + \frac{a}{m},$$

and $$y = BC\sin\theta = \frac{a\sec\theta}{m}.\sin\theta = \frac{a\tan\theta}{m},$$

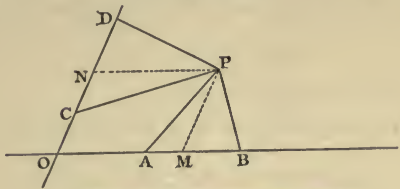
$$\therefore\ x = my + \frac{a}{m},$$

the equation to a straight line.

42. Let AB and CD be the sliding lines of lengths a and b.

Let (x, y) be co-ordinates of P.

Then area $\quad PAB = \frac{1}{2}a \times$ perpendicular from $P = \frac{1}{2}ay \sin \omega$,
and area $\quad PCD = \frac{1}{2}bx \sin \omega$,



$$\therefore\ \tfrac{1}{2}ay \sin \omega + \tfrac{1}{2}bx \sin \omega = \text{constant},$$

which is the equation to a straight line.

43. Take AB as axis of x, AC as axis of y, and let their lengths be respectively c and b.

Then the co-ordinates of B are $(c, 0)$, of C they are $(0, b)$, of F they are $(c, -c)$, of K they are $(-b, b)$.

The equation to CF is $\quad y - b = \dfrac{-c-b}{c} \cdot x.$

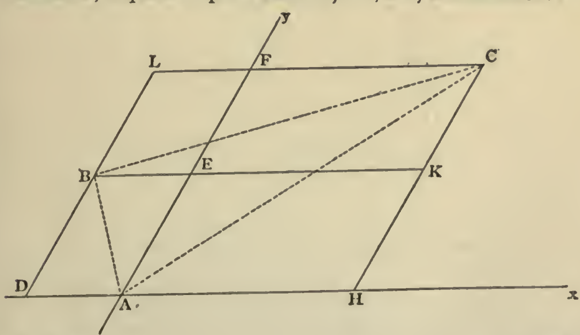
The equation to BK is $\quad y - b = \dfrac{-b}{c+b}(x+b).$

Solving these simultaneously we find they intersect at the point

$$\left(\frac{b^2c}{b^2+bc+c^2},\ \frac{bc^2}{b^2+bc+c^2}\right).$$

Also since the equation to BC is $y = \dfrac{b}{-c} \cdot x + b$, we have the equation to AL (by Art. 44) $y = \dfrac{c}{b}x$, and this is evidently satisfied by the above point of intersection.

44. Let ABC be the triangle, and let Ax, Ay, the axes, be parallel to the fixed directions; required to prove that DE, LK, FH, are concurrent.



Let the co-ordinates of B be (x_1, y_1), and of C be (x_2, y_2); therefore co-ordinates of D and E are $(x_1, 0)$ and $(0, y_1)$; therefore equation to DE is $y-y_1=-\frac{y_1}{x_1}.x$; and to FH is $y-y_2=-\frac{y_2}{x_2}.x$; and to LK is

$$y-y_2=\frac{y_1-y_2}{x_2-x_1}(x-x_1).$$

These are easily shewn to be concurrent.

45. Take O as origin, and any fixed direction through O as the initial line. Let the equations to the given lines be

$$p_1=r\cos(\theta-\alpha_1),\quad p_2=r\cos(\theta-\alpha_2),\ \&c.$$

Let the co-ordinates of X be (R, θ).

By the given equations we have

$$OA=\frac{p_1}{\cos(\theta-\alpha_1)},\quad OB=\frac{p_2}{\cos(\theta-\alpha_2)},\ \&c.$$

Hence the equation $\frac{1}{OX}=\frac{1}{OA}+\frac{1}{OB}+\ldots\ldots$

becomes
$$\frac{1}{R}=\frac{\cos(\theta-\alpha_1)}{p_1}+\frac{\cos(\theta-\alpha_2)}{p_2}+\ldots\ldots$$
$$=m\cos\theta+n\sin\theta,$$

where m and n are some constant quantities containing p_1, p_2, p_3... and α_1, α_2, α_3 ... .

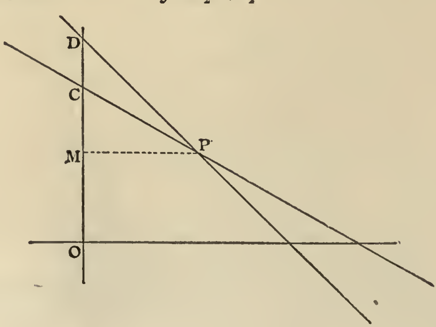
Hence
$$1=mR\cos\theta+nR\sin\theta$$
$$=mx+ny,$$

if (x, y) are the rectangular co-ordinates of X.

This locus is evidently a straight line.

46. Let PC be the line $y=m_1x+c_1$,

and PD be the line $y=m_2x+c_2$.



Hence $$OC=c_1 \text{ and } OD=c_2;$$
$$\therefore\ CD=c_2-c_1.$$

Also, solving the equations simultaneously we get the abscissa of P

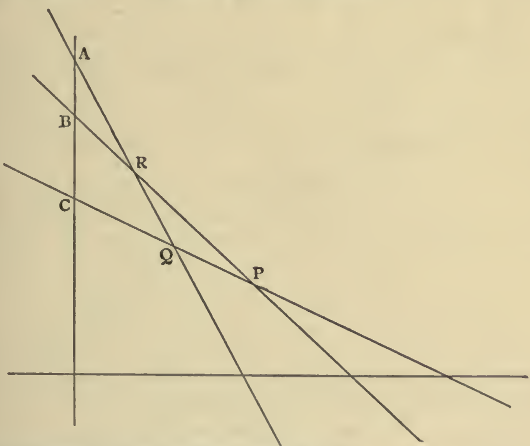
$$=\frac{c_2-c_1}{m_1-m_2}.$$

But area of $PCD=\frac{1}{2}CD\,.\,PM=\frac{1}{2}(c_2-c_1)\,.\,\frac{c_2-c_1}{m_1-m_2}$

$$=\frac{(c_2-c_1)^2}{2\,(m_1-m_2)}.$$

If the line $y=m_1x+c_1$ had been taken as crossing the axis of y above the other, the result would have been $\frac{(c_2-c_1)^2}{2\,(m_2-m_1)}$.

47. The area $$PQR=PBC+RBA-QAC,$$



and, by preceding example, this is equivalent to

$$\frac{(c_3-c_2)^2}{2\,(m_3-m_2)}+\frac{(c_2-c_1)^2}{2\,(m_2-m_1)}-\frac{(c_1-c_3)^2}{2\,(m_3-m_1)}$$
$$=\frac{(c_3-c_2)^2}{2\,(m_3-m_2)}+\frac{(c_2-c_1)^2}{2\,(m_2-m_1)}+\frac{(c_1-c_3)^2}{2\,(m_1-m_3)};$$

this may also be transformed into the shape

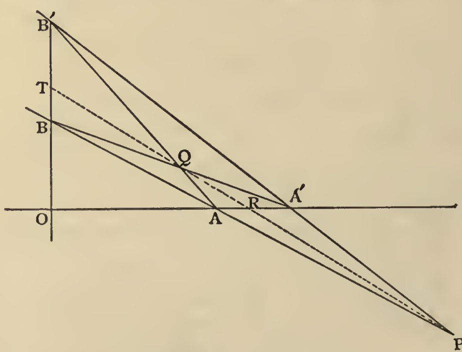
$$\frac{\{c_1\,(m_3-m_2)+c_2\,(m_1-m_3)+c_3\,(m_2-m_1)\}^2}{2\,(m_3-m_2)\,(m_2-m_1)\,(m_1-m_3)}.$$

48. In the preceding result, put $m_1=a$, $m_2=b$, $m_3=c$, $c_1=-\frac{bc}{2}$, $c_2=-\frac{ac}{2}$, $c_3=-\frac{ab}{2}$, and the result is at once obtained.

CHAPTER IV.

1. The equation $\frac{x}{a}+\frac{y}{b}-1=\frac{x}{a'}+\frac{y}{b'}-1$ evidently represents a straight line going through the intersection of the two given lines, because it is satisfied when the two given equations are *simultaneously* true; also by writing it in the shape $\frac{x}{a}+\frac{y}{b}=\frac{x}{a'}+\frac{y}{b'}$ we see that it goes through the origin. Hence it is the required line.

2. Let $OA=a$, $OB=b$, $OA'=c$, $OB'=d$.



Equation to AB is $ay+bx-ab=0$,

,, $A'B'$ is $cy+dx-cd=0$,

,, AB' is $ay+dx-ad=0$,

,, $A'B$ is $cy+bx-bc=0$.

The equation $cd(ay+bx-ab)+ab(cy+dx-cd)=0$ evidently represents a straight line through the intersection of AB and $A'B'$, that is through P; also since it can be written

$$bc(ay+dx-ad)+ad(cy+bx-bc)=0$$

it also goes through Q; hence it is the line PQ.

Putting $y=0$, we get $OR=\frac{2ac}{a+c}$, and therefore OR is an harmonic mean between a and c.

Similarly, by putting $x=0$ we get $OT=\frac{2bd}{b+d}$; hence OT is an harmonic mean between b and d.

[For a beginner, the chief difficulty in this example would be the obtaining of the equation for PQ. In this and similar cases a little judicious trying of various shapes, noticing the *forms* of the equations dealt with, will often prove successful. If however this fails, the following method will always be successful:

Since the required line goes through the intersection of AB and $A'B'$ its equation must be of the form

$$P(ay+bx-ab)+Q(cy+dx-cd)=0;$$

also as it goes through the intersection of $A'B$ and AB' it is of the form

$$R(ay+dx-ad)+S(cy+bx-bc)=0.$$

These two shapes are to be identical (Introduction, § VIII.), so that by equating coefficients we get

$$aP+cQ\equiv aR+cS;\ bP+dQ\equiv dR+bS;\ abP+cdQ\equiv adR+bcS.$$

These three equations are of course not sufficient to determine the four quantities P, Q, R, S, but they are sufficient to determine their ratios to one another, which is all that is necessary. We shall find $\frac{P}{Q}=\frac{cd}{ab}$, and $\frac{R}{S}=\frac{bc}{ad}$; hence our reason for the two forms in which the equation to PQ is written.]

3. Let CD, the bisector of the side AB, meet AB in D.

Draw DE and DF perpendicular to BC and AC.

Then if α, β, γ be the co-ordinates of *any* point in CD, we have, by similar triangles,

$$\frac{\alpha}{\beta}=\frac{DE}{DF}=\frac{DB\,.\sin B}{DA\,.\sin A}=\frac{\sin B}{\sin A};$$

therefore equation to CD is $\alpha\sin A-\beta\sin B=0$.

4. The equations to the three bisectors are

$$\alpha\sin A-\beta\sin B=0,\quad \beta\sin B-\gamma\sin C=0,\quad \gamma\sin C-\alpha\sin A=0;$$

and since any one of these equations can be obtained by adding the other two, it is evident that they are *simultaneously* true.

5. Let the perpendicular from C meet AB at N, and draw NE, NF perpendiculars to BC and AC.

Let α, β, γ be the co-ordinates of *any* point in CN, then by similar triangles

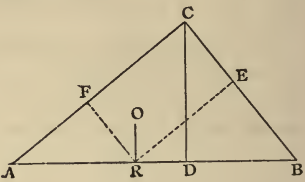
$$\frac{\alpha}{\beta}=\frac{NE}{NF}=\frac{CN\,.\sin NCE}{CN\,.\sin NCF}=\frac{\cos B}{\cos A},$$

and thus the equation to CN is

$$\alpha\cos A-\beta\cos B=0.$$

6. The equations to the three altitudes being $a\cos A - \beta\cos B = 0$, $\beta\cos B - \gamma\cos C = 0$, $\gamma\cos C - a\cos A = 0$, it is evident that any one of these equations can be derived from the other two, and therefore they are *simultaneously* true.

7. Since
$$RF = AR\sin A = \frac{c}{2}\sin A,$$
and
$$RE = RB\sin B = \frac{c}{2}\sin B,$$
therefore co-ordinates of R are
$$\left(\frac{c}{2}\sin B,\ \frac{c}{2}\sin A\right).$$



The equation to CD (by Example 5) is
$$a\cos A - \beta\cos B = 0;$$
hence (by Art. 73) equation to OR is
$$a\cos A - \beta\cos B + k = 0;$$
and since this is satisfied by the co-ordinates of R, we have
$$\frac{c}{2}\sin B\cos A - \frac{c}{2}\sin A\cos B + k = 0.$$

Subtract this equation from the previous one, and we have
$$a\cos A - \beta\cos B - \frac{c}{2}\sin B\cos A + \frac{c}{2}\sin A\cos B = 0.$$

Since, by Trigonometry, $c = \dfrac{a\sin C}{\sin A} = \dfrac{b\sin C}{\sin B}$, the second shape is soon found.

If f, g, h be the three altitudes of the triangle, we have $f = c\,.\sin B$, and so for the others; hence the equation may be written
$$(a - \tfrac{1}{2}f)\cos A - (\beta - \tfrac{1}{2}g)\cos B = 0;$$
which is generally the most convenient shape.

8. The three lines
$$(a - \tfrac{1}{2}f)\cos A - (\beta - \tfrac{1}{2}g)\cos B = 0,$$
$$(\beta - \tfrac{1}{2}g)\cos B - (\gamma - \tfrac{1}{2}h)\cos C = 0,$$
$$(\gamma - \tfrac{1}{2}h)\cos C - (a - \tfrac{1}{2}f)\cos A = 0,$$
are evidently concurrent.

9. The line $aa + b\beta = 0$ evidently goes through the point for which $a = 0$ and $\beta = 0$ simultaneously,—that is to say, through the point C; also since it may be written $aa + b\beta + c\gamma - (aa + b\beta) = 2\Delta$ (Art. 73), it reduces down to $c\gamma = 2\Delta$, or $\gamma = \dfrac{2\Delta}{c}$, which is the equation to a straight line parallel to AB.

10. The equation $a\alpha + b\beta - c\gamma = 0$ may be written as

$$a\alpha + b\beta + c\gamma - (a\alpha + b\beta - c\gamma) = 2\Delta \text{ (Art. 73)},$$

or $2c\gamma = 2\Delta = hc$, if h be the perpendicular from C on AB; this reduces to $\gamma = \frac{1}{2}h$.

Hence the straight line is parallel to AB and bisects the perpendicular from C; consequently it also bisects the sides AC and BC.

11. The given equation evidently is satisfied when $\alpha = 0$ simultaneously with $\beta \cos B - \gamma \cos C = 0$; that is to say, it is a straight line going through the foot of the perpendicular from A on BC. In like manner it is satisfied by $\beta = 0$ and $\alpha \cos A - \gamma \cos C = 0$; hence it is the equation to a straight line through the foot of the perpendicular from B on AC. Hence it is the line required.

12. This example is worked out in Art. 72.

13. Let the equation required be $fu + gv + hw = 0$; since it passes through the first given point we have $fl + gm + hn = 0$; similarly since it passes through the second given point we have $fl' + gm' + hn' = 0$. Determine the values of f and g in terms of h from these last two equations, and then substitute these values in the first equation; we get

$$u(mn' - m'n) + v(l'n - n'l) + w(lm' - ml') = 0.$$

14. The required equation will be of the shape

$$P(2au + bv + cw) + Q(bv - cw) = 0,$$

and also of the shape

$$R(2bu + av + cw) + S(av - cw) = 0.$$

But as these two shapes are to be identical (Introd. § VIII.), we have

$$2aP \equiv 2bR;\ \ bP + bQ \equiv aR + aS;\ \ cP - cQ \equiv cR - cS.$$

Hence we get $P = -Q\dfrac{b}{2a+b}$, so that our equation is

$$b(2au + bv + cw) - (2a + b)(bv - cw) = 0;$$

which reduces to $$ab(u - v) + (ac + bc)w = 0.$$

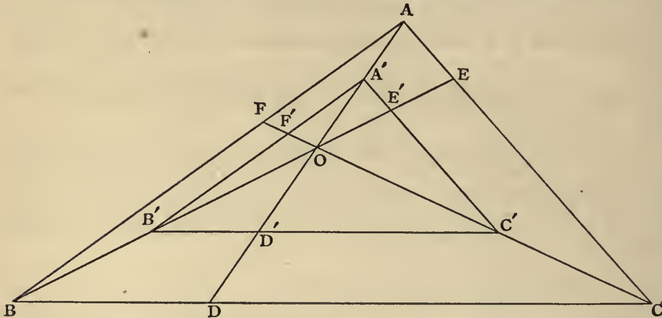
15. Let R, r be the radii of the circumscribed and inscribed circles.

Then for the inscribed centre we have $\alpha = \beta = \gamma = r$; for the circumscribed centre we have $\alpha = R\cos A$; $\beta = R\cos B$; $\gamma = R\cos C$.

Both these sets of co-ordinates satisfy the given equation, and it is therefore the equation required.

16. The equation $cv - bw = 0$ is satisfied, when $v = 0$ and $w = 0$, and also when $v = b$, and $w = c$, hence it goes through the two points A and A'; that is to say it is the equation to AA'.

Similarly the equation to BB' is $aw-cu=0$, and to CC' is $bu-av=0$.



Writing these equations in the shapes $\frac{v}{b}-\frac{w}{c}=0$, $\frac{w}{c}-\frac{u}{a}=0$, $\frac{u}{a}-\frac{v}{b}=0$, we see that the three lines are concurrent.

17. As in the preceding example, the equations to AA', BB', CC' are $\frac{v}{b}-\frac{w}{c}=0$, $\frac{w}{c}-\frac{u}{a}-0$, $\frac{u}{a}-\frac{v}{b}=0$.

Now the equation $\frac{v}{b}-\frac{w}{c}+\frac{u}{a}=0$ evidently represents a line going through the intersection of AA' and BC, that is through D; it also represents a line through the intersection of BB' and AC, that is through E; hence it is the line DE.

Similarly the equation to EF is $\frac{v}{b}+\frac{w}{c}-\frac{u}{a}=0$, and to DF is

$$\frac{u}{a}+\frac{w}{c}-\frac{v}{b}=0.$$

Adding the three equations together, we get the equation

$$\frac{u}{a}+\frac{v}{b}+\frac{w}{c}=0;$$

consequently this equation is true when the other three are simultaneously true,—that is to say, they all intersect on this straight line.

The equation to $D'E'$ will be easily seen to be $\frac{v}{b}+\frac{u}{a}-\frac{w}{c}=1$, and so for the others. And the three lines will evidently all meet on the line

$$\frac{u}{a}+\frac{v}{b}+\frac{w}{c}=3.$$

[*Note.* If any difficulty was experienced in finding by inspection the equations to DE &c., recourse could be had to the method detailed in the note to Example 2.]

18. Since the equation to FG is

$$lu - 2mv + nw = 0,$$

and to BA is $u = 0$, it is evident that the equation

$$lu - (lu - 2mv + nw) = 0$$

represents a straight line through their intersection, that is through P; also, as it may be written $2mv - nw = 0,$

it evidently goes through C; hence it is the line CP.

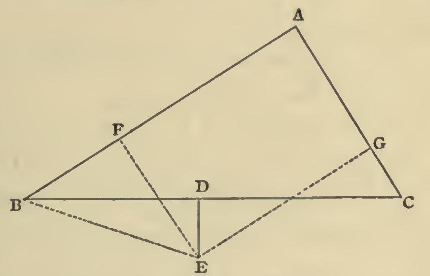
Similarly the equation to DP is

$$2lu - 2mv + nw = 0;$$

and to AQ is $lu - 2mv + 2nw = 0;$

and to BQ is $lu - 2mv = 0.$

19. Let DE be one of the three perpendiculars drawn outwards; then since each perpendicular is proportional to the corresponding side, it follows that the angle DBE and the two similar angles for the other perpendiculars will be all equal; let them be denoted by θ.



Then $EF = BE \,.\, \sin(B+\theta) = DE \,.\, \operatorname{cosec}\theta \,.\, \sin(B+\theta).$

Similarly $EG = DE \,.\, \operatorname{cosec}\theta \,.\, \sin(C+\theta).$

Hence the equation to AE is

$$\frac{\gamma}{\beta} = \frac{\sin(B+\theta)}{\sin(C+\theta)},$$

or $$\gamma \,.\, \sin(C+\theta) - \beta \sin(B+\theta) = 0.$$

Similarly the other two corresponding lines are

$$\alpha \,.\, \sin(A+\theta) - \gamma \,.\, \sin(C+\theta) = 0, \text{ and } \beta \sin(B+\theta) - \alpha \,.\, \sin(A+\theta) = 0,$$

and these are obviously concurrent.

If the lines be drawn inwards, we have only to write $-\theta$ for θ in the previous results.

20. In the figure to Art. 75, let FG meet BA in P, and CD in Q; and let BD meet EF in H. Also let CA and FE be produced to meet in K.

Then it is required to prove that GE, GF, HA, HC, KB, KD meet three and three in four points.

Their equations are given in Art. 75, and are as follows:

to AH we have	$2lu - mv + nw = 0$;	to CH we have	$mv + nw = 0$;
to KD ,, ,,	$lu - mv + 2nw = 0$;	to KB ,, ,,	$lu + mv = 0$;
to GF ,, ,,	$lu - 2mv + nw = 0$;	to GE ,, ,,	$lu - nw = 0$.

It is evident from these equations that AH, KB, GF, are concurrent; and AH, KD, GE; and CH, KD, GF; and CH, KB, GE; and it is easily seen that no other group of three lines is concurrent.

21. Let us take the corner C; then it is required to prove that if BC and DC be the directions of the sides of a parallelogram, and CA the direction of one diagonal, then HC will be the direction of the other diagonal.

Draw AR and AT parallel to BC and DC to meet DC and BC; it is then evidently required to prove that HC is parallel to RT.

The equation to BC is $v=0$; hence we may take the equation to AR as $v+k=0$. (Art. 73.)

Also equation to CD is $w=0$.

Now the equation $mk+nw=0$ represents a straight line parallel to CD; also since it can be written

$$m(v+k)-(mv-nw)=0,$$

it is a line through the intersection of AR and AC; hence it must be the line AT.

Again, the equation $mv+mk+nw=0$ represents a straight line through the intersection of AR and CD since it can be written

$$m(v+k)+nw=0;$$

and it is also a straight line through the intersection of AT and CB; hence it is the line RT.

Also this line $mv+mk+nw=0$ is evidently parallel to $mv+nw=0$, that is, to HC.

Similarly we may treat the other corners of the quadrilateral.

22. The given equation may be written

$$\alpha \sin A \,.\, \cos(B+C) \,.\, \sin(B-C) + \beta \sin B \,.\, \cos(A+C) \,.\, \sin(C-A)$$
$$+ \gamma \sin C \,.\, \cos(A+B) \,.\, \sin(A-B) = 0,$$

or $\alpha \sin A \,.\, \{\sin 2B - \sin 2C\} + \beta \sin B \,.\, \{\sin 2C - \sin 2A\}$

$$+ \gamma \sin C \,.\, \{\sin 2A - \sin 2B\} = 0.$$

For the point mentioned in Ex. 4 we have

$$\alpha \sin A = \beta \sin B = \gamma \sin C,$$

so that in this case the above equation reduces to

$$\sin 2B - \sin 2C + \sin 2C - \sin 2A + \sin 2A - \sin 2B = 0,$$

which is evidently true.

Again, the given equation can be written

$$\alpha \cos A \,.\, \sin (B+C) \,.\, \sin (B-C) + \beta \cos B \,.\, \sin (A+C) \,.\, \sin (C-A) + \gamma \cos C \,.\, \sin (A+B) \,.\, \sin (A-B) = 0,$$

or $$\alpha \cos A \,.\, \{\sin^2 B - \sin^2 C\} + \beta \cos B \,.\, \{\sin^2 C - \sin^2 A\} + \gamma \cos C \,.\, \{\sin^2 A - \sin^2 B\} = 0.$$

For the point mentioned in Ex. 6 we have

$$\alpha \cos A = \beta \cos B = \gamma \cos C,$$

so that in this case the equation reduces to

$$\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B = 0,$$

which is evidently true.

Lastly, the equations of Ex. 8 may be written

$$\alpha \cos A - \beta \cos B = \frac{c}{2} \sin (B-A),$$

and $$\beta \cos B - \gamma \cos C = \frac{a}{2} \sin (C-B),$$

and $$\gamma \cos C - \alpha \cos A = \frac{b}{2} \,.\, \sin (A-C).$$

Also the given equation may be transformed as before into

$$\alpha \,.\, \cos A \,.\, \{\sin^2 B - \sin^2 C\} + \beta \cos B \,.\, \{\sin^2 C - \sin^2 A\} + \gamma \cos C \, \{\sin^2 A - \sin^2 B\} = 0,$$

from which we get

$$\sin^2 C \, \{\alpha \cos A - \beta \cos B\} + \sin^2 A \, \{\beta \cos B - \gamma \cos C\} + \sin^2 B \, \{\gamma \cos C - \alpha \cos A\} = 0;$$

and when the equations of Ex. 8 are true this becomes

$$\frac{c}{2} \sin^2 C \,.\, \sin (B-A) + \frac{a}{2} \sin^2 A \,.\, \sin (C-B) + \frac{b}{2} \,.\, \sin^2 B \,.\, \sin (A-C) = 0.$$

But since

$$c : a : b :: \sin C : \sin A : \sin B :: \sin (A+B) : \sin (B+C) : \sin (A+C),$$

the equation can be transformed into

$$\sin^2 C \,.\, \sin (A+B) \sin (B-A) + \sin^2 A \,.\, \sin (B+C) \sin (C-B) + \sin^2 B \,.\, \sin (A+C) \,.\, \sin (A-C) = 0,$$

or

$$\sin^2 C \,.\, (\sin^2 B - \sin^2 A) + \sin^2 A \, (\sin^2 C - \sin^2 B) + \sin^2 B \, (\sin^2 A - \sin^2 C) = 0,$$

which is evidently true.

22. (*Aliter.*) The co-ordinates of the point named in Ex. 4 can be found by Trigonometry to be

$$\tfrac{1}{3} b \sin C, \ \tfrac{1}{3} c \sin A, \ \tfrac{1}{3} a \sin B,$$

and these will satisfy the given equation.

Again, the co-ordinates of the point in Ex. 6 are by Trigonometry

$$\frac{c}{\sin C} \cdot \cos B \cdot \cos C, \quad \frac{c}{\sin C} \cdot \cos C \cdot \cos A, \quad \frac{c}{\sin C} \cdot \cos A \cdot \cos B,$$

and these satisfy the equation.

Again, the co-ordinates of the point in Ex. 8 are by Trigonometry

$$R \cos A, \quad R \cos B, \quad R \cos C,$$

and these satisfy the equation.

23. Since the line AP passes through the intersection of $v=0$ and $w=0$, it can be represented by $mv-nw=0$, if m and n be suitably chosen.

Again, BP can in like manner be represented by $nw-lu=0$.

Hence CP *must* be represented by

$$(mv-nw)+(nw-lu)=0,$$

that is, by $mv-lu=0$; because it passes through the intersection of AP and BP, and through the intersection of BC and AC.

Again, the equation

$$nw-lu+mv=0$$

represents a straight line through the intersection of BP and AC, and through the intersection of CP with AB; hence it is the line EF.

Similarly the equation to FD is

$$lu-mv+nw=0,$$

and to DE is

$$mv-nw+lu=0.$$

24. It is evident that the equation

$$lu+mv+nw=0$$

is satisfied when

$$nw-lu+mv=0$$

simultaneously with $u=0$; hence this straight line passes through A'. Similarly it passes through B' and C'.

25. BB' passes through the intersection of $u=0$ with $w=0$, and through the intersection of $v=0$ with

$$lu-mv+nw=0;$$

hence its equation will be

$$(lu-mv+nw)+mv=0,$$

or

$$lu+nw=0.$$

Similarly the equation to CC' is

$$mv+lu=0,$$

and the equation to AD is

$$mv-nw=0;$$

and these three equations are evidently satisfied simultaneously; hence the lines are concurrent.

Similarly for the other groups.

26. It is evident that the lines AB, BC, CA bisect the exterior angles of the triangle $A'B'C'$. Hence if we denote the sides of $A'B'C'$ by the equations $\alpha=0$, $\beta=0$, $\gamma=0$, the equations to BC, CA, AB will (by Art. 72) be

$$\beta+\gamma=0,\quad \gamma+\alpha=0,\quad \alpha+\beta=0.$$

Now since AA' goes through the intersection of $\beta=0$ with $\gamma=0$, and of $\gamma+\alpha=0$ with $\alpha+\beta=0$, its equation is evidently $\beta-\gamma=0$.

Hence it is the bisector of the internal angle $B'A'C'$, and is consequently perpendicular to BC the bisector of the external angle.

Similarly for the others.

27. The line EDF goes through the intersection of $\beta-\gamma=0$ with $\alpha=0$ (Art. 72); hence its equation will be $\beta-\gamma-l\alpha=0$, where l is some undetermined constant.

The point O is the intersection of $\beta-\gamma=0$ with $\alpha-\gamma=0$; hence the equation to OF is evidently $\beta-\gamma-l(\alpha-\gamma)=0$, as this also goes through the intersection of $\beta-\gamma-l\alpha=0$ with $\gamma=0$.

Again, the point O' is the intersection of $\beta-\gamma=0$ with $\alpha+\beta=0$; hence the equation to $O'E$ is $\beta-\gamma-l(\alpha+\beta)=0$, since this also goes through the intersection of $\beta-\gamma-l\alpha=0$ with $\beta=0$.

Subtract the equation to OF from that to $O'E$ and we get $\beta+\gamma=0$; hence OF and $O'E$ intersect on the bisector of the external angle at A (Art. 72); and this bisector is perpendicular to AO.

In like manner OE and $O'F$ intersect on the same straight line.

28. If from any point on either bisector perpendiculars be drawn to the given straight lines, then the one perpendicular will be equal to the other.

Hence (by Art. v. of Chap. IV.) we get the equation

$$\frac{l\alpha+m\beta+n\gamma}{\sqrt{(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)}}$$
$$=\pm\frac{l'\alpha+m'\beta+n'\gamma}{\sqrt{\{l'^2+m'^2+n'^2-2m'n'\cos A-2n'l'\cos B-2l'm'\cos C\}}},$$

where the positive sign belongs to one bisector, and the negative to the other.

29. Let each of the fractions in the question be equal to p, so that

$$\alpha=p\,(m'n''-m''n'),$$
$$\beta=p\,(n'l''-n''l'),$$
$$\gamma=p\,(l'm''-l''m'),$$
$$2\Delta=p\,\{a\,(m'n''-m''n')+b\,(n'l''-n''l')+c\,(l'm''-l''m')\}.$$

Now it will be found that the above values will satisfy the equation $l'\alpha+m'\beta+n'\gamma=0$ and $l''\alpha+m''\beta+n''\gamma=0$, and will also satisfy the invariable condition $a\alpha+b\beta+c\gamma=2\Delta$ (see Art. 73); consequently the above equations do fix the co-ordinates of the intersection of the two given lines.

30. Taking f, g, h as the three co-ordinates determined by the equations in the last example, it follows that length of the required perpendicular is (by Art. v. of Chap. IV.)

$$\frac{lf+mg+nh}{\sqrt{(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)}}.$$

31. This is fully worked out in the Answers.

32. Parallel straight lines may be regarded as lines intersecting on the straight line at infinity. (See Art. VIII. of Chap. IV.)

Hence the given equations and the equation $a\alpha+b\beta+c\gamma=0$ must be *simultaneously* true.

Let each of the quantities $\frac{\alpha}{\lambda}$, $\frac{\beta}{\mu}$, $\frac{\gamma}{\nu}$ be equal to k, so that

$$\alpha=\lambda k;\quad \beta=\mu k;\quad \gamma=\nu k.$$

Substitute these values in $a\alpha+b\beta+c\gamma=0$, and we get as our required condition $$a\lambda+b\mu+c\nu=0.$$

33. The first equation in the last example may be written $\mu\alpha-\lambda\beta=0$; and the general form of the equation to a line parallel to this is (by Art. 73)

$$\mu\alpha-\lambda\beta+p\,(a\alpha+b\beta+c\gamma)=0,$$

where p is some constant.

This equation may be written

$$\alpha+\beta\,\frac{bp-\lambda}{\mu+ap}+\gamma\,\frac{pc}{\mu+ap}=0;$$

and if it is to be identical with the given equation, which may be written

$$\alpha+\frac{m}{l}\beta+\frac{n}{l}\,\gamma=0,$$

we must (by Introd. § VIII.) have

$$\frac{bp-\lambda}{\mu+ap}\equiv\frac{m}{l},\ \text{and}\ \frac{pc}{\mu+ap}\equiv\frac{n}{l}.$$

Eliminate p (by finding the value of p from each of these equations, and equating the two values so found), and the resulting equation will be the required condition.

This condition will be found to be

$$cl\lambda+cm\mu-n(a\lambda+b\mu)=0.$$

But by the last example we have

$$a\lambda+b\mu=-c\nu.$$

Hence the condition becomes

$$cl\lambda+cm\mu+cn\nu=0,\quad\text{or}\quad l\lambda+m\mu+n\nu=0.$$

33. (*Aliter.*) If the three lines in the previous example are to be parallel to $l\alpha+m\beta+n\gamma=0$, then the three given equations and the equation $l\alpha+m\beta+n\gamma=0$ must be simultaneously true for the same point on the line at infinity. These three equations give us $\alpha=q\lambda$, $\beta=q\mu$, $\gamma=q\nu$, where q is some constant.

Substitute these in the equation $l\alpha+m\beta+n\gamma=0$, and we get the condition

$$l\lambda+m\mu+n\nu=0.$$

34. Let each of the given fractions be equal to p, so that

$$\alpha=\alpha'+\lambda p;\quad \beta=\beta'+\mu p;\quad \gamma=\gamma'+\nu p;$$

$$\therefore\quad a\alpha+b\beta+c\gamma=a\alpha'+b\beta'+c\gamma'+p(\lambda a+\mu b+\nu c).$$

But if α, β, γ and α', β', γ' are co-ordinates of points, we must (by Art. 73) have $a\alpha+b\beta+c\gamma=2\Delta$ and $a\alpha'+b\beta'+c\gamma'=2\Delta$. Hence our equation above becomes $\quad p(\lambda a+\mu b+\nu c)=0,\quad$ or $\quad \lambda a+\mu b+\nu c=0,$

which is the required condition.

35. In the last example we obtained the condition $p(\lambda a+\mu b+\nu c)=0$. If $\lambda a+\mu b+\nu c$ is not $=0$, then the above equation can only be satisfied by $p=0$.

In this case we have $\alpha=\alpha'$, $\beta=\beta'$, $\gamma=\gamma'$, or in other words the variable co-ordinates (α, β, γ) are only applicable to the fixed point $(\alpha', \beta', \gamma')$, so that the required locus is this point.

36. Draw AN and ER perpendicular to BC.

Now since E is the intersection of $n\gamma-l\alpha=0$ with $\beta=0$, it follows that if we solve these two equations simultaneously with the invariable condition $a\alpha+b\beta+c\gamma=2\Delta$, we shall get the co-ordinates of E.

We get α or $\quad ER=\dfrac{2n\Delta}{na+lc}.$

Also $AN\times a=2\Delta$, or $\quad AN=\dfrac{2\Delta}{a};$

$$\therefore\quad \frac{ER}{AN}=\frac{na}{na+lc};\text{ but by similar triangles }\frac{ER}{AN}=\frac{EC}{AC};$$

$$\therefore\quad \frac{EC}{AC}=\frac{na}{na+lc},\quad \therefore\quad \frac{AE}{AC}=\frac{lc}{na+lc}.$$

Similarly $\quad \dfrac{AF}{AB}=\dfrac{lb}{ma+lb}.$

But $\quad \dfrac{\text{area of }AEF}{\text{area of }ABC}=\dfrac{\frac{1}{2}AE\,.\,AF\,.\,\sin A}{\frac{1}{2}AC\,.\,AB\,.\,\sin A}=\dfrac{AE}{AC}\times\dfrac{AF}{AB}=\dfrac{l^2bc}{(na+lc)(ma+lb)}.$

Similarly $\quad \dfrac{\text{area of }BDF}{\text{area of }ABC}=\dfrac{m^2ac}{(nb+mc)(lb+ma)},$

and $\quad \dfrac{\text{area of }CDE}{\text{area of }ABC}=\dfrac{n^2ab}{(lc+na)(mc+nb)};$

$$\therefore \frac{\text{area of } DEF}{\text{area of } BAC} = 1 - \frac{l^2bc}{(na+lc)(ma+lb)} - \frac{m^2ac}{(nb+mc)(lb+ma)}$$
$$- \frac{n^2ab}{(lc+na)(mc+nb)} = \frac{2abclmn}{(na+lc)(ma+lb)(nb+mc)}.$$

37. It is evident by Art. 76 that the triangles are homologous, the orthocentre of the original triangle being the centre of homology.

Then this question is a particular case of Examples 23, 24, and consequently the required equation can be got by putting $\cos A$, $\cos B$, $\cos C$ for l, m, n.

Or it may be treated independently thus: let the perpendiculars be AD, BE, CF, meeting at the orthocentre P.

Then the equation to BC is $\alpha = 0$, and the equation to EF is (by Ex. 11) $\beta \cos B + \gamma \cos C - \alpha \cos A = 0$; and these two straight lines evidently intersect on the straight line $\alpha \cos A + \beta \cos B + \gamma \cos C = 0$.

Similarly the other sides intersect on this straight line, which is therefore the axis of homology.

38. Writing the equation as a quadratic in α, we have

$$A\alpha^2 + 2\alpha(E\gamma + F\beta) + B\beta^2 + C\gamma^2 + 2D\beta\gamma = 0,$$

but we wish this equation to have rational roots, and therefore (by Introd. § III.) we see that

$$(E\gamma + F\beta)^2 - A(B\beta^2 + C\gamma^2 + 2D\beta\gamma)$$

must be a perfect square.

Writing this in the shape

$$\gamma^2\{E^2 - AC\} + 2\beta\gamma\{EF - AD\} + \beta^2\{F^2 - AB\},$$

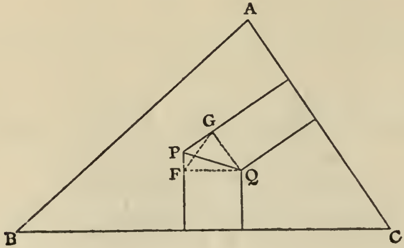
we know (by Introd. § I.) that we must have

$$(EF - AD)^2 = (E^2 - AC)(F^2 - AB),$$

which easily reduces to

$$AD^2 + BE^2 + CF^2 - ABC - 2DEF = 0.$$

39. We shall use p, q, r as an abbreviation for $\alpha_1 - \alpha_2$, $\beta_1 - \beta_2$, $\gamma_1 - \gamma_2$.



Let P and Q be the two given points, so that

$$PF = \alpha_1 - \alpha_2 = p,$$

and

$$PG = \beta_1 - \beta_2 = q.$$

Now PQ is evidently the diameter of the circle passing through the points PGF, and therefore, by Trigonometry,

$$PQ = \frac{FG}{\sin FPG} = \frac{FG}{\sin C};$$

$$\therefore \quad PQ^2 = \frac{FG^2}{\sin^2 C} = \frac{p^2 + q^2 - 2pq \cos FPG}{\sin^2 C} = \frac{p^2 + q^2 + 2pq \cos C}{\sin^2 C},$$

and this is the first shape required.

Secondly, since we have $a\alpha_1 + b\beta_1 + c\gamma_1 = 2\Delta$, and $a\alpha_2 + b\beta_2 + c\gamma_2 = 2\Delta$, by subtraction we get $pa + qb + rc = 0$, which is equivalent to

$$p \sin A + q \sin B + r \sin C = 0.$$

Hence

$$p \sin A + q \sin B = -r \sin C.$$

Squaring this equation, and transposing, we get

$$2pq \sin A \sin B = r^2 \sin^2 C - p^2 \sin^2 A - q^2 \sin^2 B.$$

But from our first result

$$PQ^2$$

$$= \frac{p^2 \sin A \,.\, \sin B + q^2 \sin A \,.\, \sin B + 2pq \sin A \,.\, \sin B \,.\, \cos C}{\sin A \,.\, \sin B \,.\, \sin^2 C}$$

$$= \frac{p^2 \sin A \,.\, \sin B + q^2 \sin A \,.\, \sin B + r^2 \sin^2 C \,.\, \cos C - p^2 \sin^2 A \,.\, \cos C - q^2 \sin^2 B \,.\, \cos C}{\sin A \,.\, \sin B \,.\, \sin^2 C}$$

$$= \frac{p^2 \sin A \,.\, \{\sin(A+C) - \sin A \,.\, \cos C\} + q^2 \sin B \{\sin(B+C) - \sin B \,.\, \cos C\} + r^2 \sin^2 C \cos C}{\sin A \,.\, \sin B \,.\, \sin^2 C}$$

$$= \frac{p^2 \sin A \,.\, \cos A \,.\, \sin C + q^2 \sin B \,.\, \cos B \,.\, \sin C + r^2 \sin^2 C \,.\, \cos C}{\sin A \,.\, \sin B \,.\, \sin^2 C}.$$

Multiply numerator and denominator by $\dfrac{2}{\sin C}$, and we get

$$\frac{p^2 \sin 2A + q^2 \sin 2B + r^2 \sin 2C}{2 \sin A \,.\, \sin B \,.\, \sin C},$$

as required.

Thirdly, we have proved above that

$$p = -\frac{r \sin C + q \sin B}{\sin A}, \text{ so that } p^2 = -\frac{pr \sin C + pq \sin B}{\sin A}.$$

Obtaining a similar value for q^2, and substituting in the first result, we at once get the third result.

40. Let the distance between the points (α, β, γ) and $(\alpha', \beta', \gamma')$ be called x.

Then, by hypothesis,

$$\frac{\alpha-\alpha'}{\lambda}=x, \text{ or } \alpha-\alpha'=\lambda x.$$

Similarly, $\beta-\beta'=\mu x$, and $\gamma-\gamma'=\nu x$.

Now, by Example 39, we have

$$x^2=\frac{(\alpha-\alpha')^2 \sin 2A+(\beta-\beta')^2 \sin 2B+(\gamma-\gamma')^2 \sin 2C}{2 \sin A \,.\, \sin B \,.\, \sin C},$$

which becomes

$$x^2=\frac{\lambda^2 x^2 \sin 2A+\mu^2 x^2 \sin 2B+\nu^2 x^2 \sin 2C}{2 \sin A \,.\, \sin B \,.\, \sin C};$$

clear this equation of fractions, and divide by x^2, and we obtain

$$\lambda^2 \sin 2A+\mu^2 \sin 2B+\nu^2 \sin 2C=2 \sin A \,.\, \sin B \,.\, \sin C.$$

CHAPTER V.

1. $$r^2=a^2 \cos 2\theta=a^2 (\cos^2\theta-\sin^2\theta).$$

Multiply both sides by r^2, and we get

$$r^4=a^2 (r^2 \cos^2\theta-r^2 \sin^2\theta),$$

and by Art. 8 this becomes

$$(x^2+y^2)^2=a^2 (x^2-y^2).$$

2. Let x_1 and y_1 be the new co-ordinates, and x, y the old ones; θ the angle turned through.

Now $\tan\theta=2$, therefore $\sin\theta=\frac{2}{\sqrt{5}}$, and $\cos\theta=\frac{1}{\sqrt{5}}$.

Hence, by Art. 81, $$x=(x_1-2y_1)\,.\,\frac{1}{\sqrt{5}},$$

and $$y=(2x_1+y_1)\,.\,\frac{1}{\sqrt{5}}.$$

Substituting in the given equation we get

$$x_1^2-4y_1^2=a^2.$$

3. Using the same co-ordinates as in previous example, we have

$$x=(x_1-y_1)\frac{1}{\sqrt{2}}, \text{ and } y=(x_1+y_1)\frac{1}{\sqrt{2}}.$$

Now the given equation can be transformed to the shape

$$4xy=(c-x-y)^2,$$

and substituting we at once obtain

$$y_1^2=\sqrt{2}cx_1-\frac{c^2}{2}.$$

4. With the same notation, by Art. 83, we have

$$x=x_1+y_1\cos\alpha, \text{ and } y=y_1\sin\alpha;$$

substituting, we get

$$y_1^2\sin^2\alpha+4ay_1\sin\alpha\cot\alpha-4ax_1-4ay_1\cos\alpha=0, \text{ or } y_1^2\sin^2\alpha=4ax_1.$$

5. Here $$x=(x_1-y_1)\frac{1}{\sqrt{2}}, \text{ and } y=(x_1+y_1)\frac{1}{\sqrt{2}}.$$

Substituting, we get

$$x_1^4-2x_1^2y_1^2+y_1^4-2\sqrt{2}ax_1^3-6\sqrt{2}ax_1y_1^2=0;$$

and as the second and fourth powers of y_1 are the only ones that occur, it is evident that the equation can be solved with respect to y_1.

6. By Art. 83 we have

$$m=\frac{\sin(\omega-\alpha)}{\sin\omega}, \text{ and } m'=\frac{\sin\alpha}{\sin\omega}.$$

Hence $$m=\frac{\cos\alpha\,.\,\sin\omega-\sin\alpha\,.\,\cos\omega}{\sin\omega}=\cos\alpha-m'\cos\omega;$$

$$\therefore\; m+m'\cos\omega=\cos\alpha.$$

Also $$m'\sin\omega=\sin\alpha.$$

Square these two results and add; we get

$$m^2+m'^2+2mm'\cos\omega=1,$$

or $$m^2+m'^2-1=-2mm'\cos\omega.$$

Similarly, since $$n=\frac{\sin(\omega-\beta)}{\sin\omega}, \text{ and } n'=\frac{\sin\beta}{\sin\omega},$$

we get $$n^2+n'^2-1=-2nn'\cos\omega.$$

Hence $$\frac{m^2+m'^2-1}{n^2+n'^2-1}=\frac{mm'}{nn'}.$$

CHAPTER VI.

1. The first equation may be written

$$(x-2)^2+(y+2)^2=9.$$

Hence the centre is $(2, -2)$, and the radius is 3.

The second equation may be written

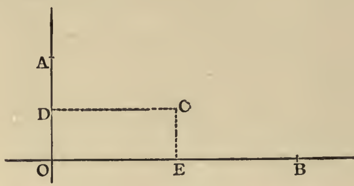
$$(x+3)^2+(y-\tfrac{3}{2})^2=\tfrac{49}{4}.$$

Hence the centre is $(-3, \tfrac{3}{2})$, and the radius is $\tfrac{7}{2}$.

2. Solving the equations $x+y=-1$ and $x^2+y^2=25$ simultaneously, we find that the first straight line meets the circle at the two points $(3, -4)$ and $(-4, 3)$. Similarly the second line meets it at the points $(0, -5)$ and $(-5, 0)$.

In the third case, the solution of the two equations gives two identical values of x, namely -4; and consequently there are two identical values of y, namely -3; hence the line *touches* the circle at $(-4, -3)$.

3. Let $OB=h$, and $OA=k$.



Let C be the centre of the circle, OE, CE its co-ordinates. Then, by Euclid III. 3,

$$OE=\frac{h}{2}, \quad CE\frac{k}{2}.$$

Also $$OC^2=OE^2+CE^2=\frac{h^2+k^2}{4}.$$

Hence the circle is $$\left(x-\frac{h}{2}\right)^2+\left(y-\frac{k}{2}\right)^2=\frac{h^2+k^2}{4},$$

which reduces to $$x^2+y^2-hx-ky=0.$$

3. (*Aliter.*) The equation to *any* circle through the origin must be of the form $x^2+y^2+Px+Qy=0$, because it must be satisfied by $x=0$ and $y=0$ simultaneously.

But if this circle is to pass through the point $(h, 0)$, we get

$$h^2+Ph=0, \quad \text{or} \quad P=-h.$$

Similarly, if it goes through the point $(0, k)$, we get

$$k^2+Qk=0, \quad \text{or} \quad Q=-k.$$

Hence the required equation is

$$x^2+y^2-hx-ky=0.$$

4. The line joining (h, k) and (h', k') will be a chord of the circle; hence (by Euc. III. 1) the centre of the circle will be in the straight line bisecting this chord at right angles.

The middle point of the chord is $\left(\frac{h+h'}{2}, \frac{k+k'}{2}\right)$, and the equation to the chord is $y-k=\frac{k'-k}{h'-h}(x-h)$.

The equation to a line perpendicular to this chord and passing through the middle point of the chord is (Art. 44)

$$y-\frac{k+k'}{2}=-\frac{h'-h}{k'-k}\left(x-\frac{h+h'}{2}\right),$$

which reduces to the given shape.

5. The co-ordinates of the centre will be $\left(\dfrac{x'+x''}{2}, \dfrac{y'+y''}{2}\right)$, and the length of the radius will be half the length of the line joining (x', y') and (x'', y''), that is to say it will be $\frac{1}{2}\sqrt{(x'-x'')^2+(y'-y'')^2}$.

Hence the circle is

$$\left(x-\frac{x'+x''}{2}\right)^2+\left(y-\frac{y'+y''}{2}\right)^2=\frac{(x'-x'')^2+(y'-y'')^2}{4},$$

which becomes $x^2+y^2-x(x'+x'')-y(y'+y'')+x'x''+y'y''=0$.

6. Take the middle point of AB as origin, and AB as axis of x. Let $AB=2a$; let (x, y) be the co-ordinate of P.

Then (by Art. 9), $\qquad AP^2=(a+x)^2+y^2,$

and $\qquad BP^2=(a-x)^2+y^2.$

But $AP=m\,.\,BP$;

$$\therefore\ (a+x)^2+y^2=m^2\{(a-x)^2+y^2\},$$

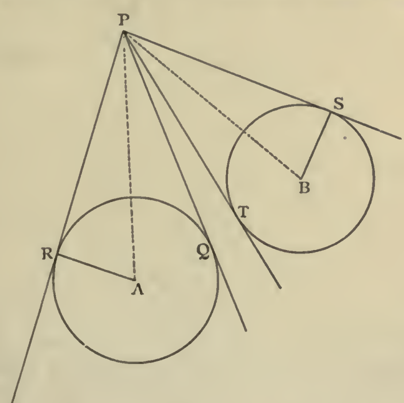
or

$$(1-m^2)\,x^2+(1-m^2)\,y^2+2ax\,(1+m^2)=(m^2-1)\,a^2.$$

Now if $1-m^2$ is not zero we can divide by it, and we shall then have the usual form of the equation to a circle.

If $m=1$, the equation reduces to $4ax=0$, or $x=0$; and this is a straight line bisecting AB at right angles.

7. If the angle $QPR=SPT$, it follows that $APR=BPS$. Hence by similar triangles $AP:BP::AR:BS$, which is a constant ratio. Hence the



problem is identical with the previous. [If $AR=BS$, so that $AP=BP$, the focus becomes the straight line called the *Radical Axis*.]

8. Eliminating y between the two given equations, we get

$$x^2\left(\frac{h^2+k^2}{h^2}\right)+2x\left\{\frac{k}{h}(b-k)-a\right\}+k^2-2bk=0.$$

Similarly by eliminating x we get a quadratic to find y.

If the straight line *touches* the circle, the two values of x obtained must be equal; hence (Introd. § I.) we have

$$\left\{\frac{k}{h}(b-k)-a\right\}^2=\left(\frac{h^2+k^2}{h^2}\right)(k^2-2bk).$$

This reduces to the form given in the Answers.

9. Let $y=mx$ be the tangent; it is necessary to determine m.

Solving this equation simultaneously with the given equation we get

$$x^2(1+m^2)-x(2m+3)=0.$$

As this is to have equal roots, the line being a *tangent*, we have (Introd. § I.)

$$(2m+3)^2=(1+m^2)\times 0, \text{ or } m=-\tfrac{3}{2}.$$

Hence the tangent required is

$$y=-\tfrac{3}{2}x, \text{ or } 2y+3x=0.$$

10. The equation

$$\{(x-a)^2+(y-b)^2-c^2\}-\{(x-b)^2+(y-a)^2-c^2\}=0$$

is evidently satisfied when the two given equations are *simultaneously* true, and therefore represents *some* line passing through the intersections of the two circles; and since it reduces to $x-y=0$ it is the equation to a straight line. Hence it represents the common chord, and is a line in which every point has its ordinate equal to its abscissa.

Consequently we may take (h, h) and (k, k) as the two common points of the circles, and these will be the roots of the equation

$$(x-a)^2+(x-b)^2=c^2;$$

$$\therefore\ h+k=a+b, \text{ and } hk=\frac{a^2+b^2-c^2}{2}.$$

Now the length of the common chord is

$$\sqrt{(h-k)^2+(h-k)^2},$$

or $$\sqrt{2(h+k)^2-8hk},$$ (see Introd. § V.)

or $$\sqrt{2(a+b)^2-4(a^2+b^2-c^2)},$$

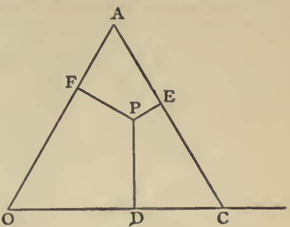
or $$\sqrt{4c^2-2(a-b)^2}.$$

11. Let (x, y) be the moving point P; take a as length of a side of the square, and let two of the sides of the square be our axes.

Then the distances of P from the sides of the square will be x, y, $x\sim a$, $y\sim a$; hence the locus of P is $x^2+y^2+(x-a)^2+(y-a)^2=\text{constant}=c^2$ suppose.

This becomes $x^2+y^2-ax-ay+a^2-\dfrac{c^2}{2}=0$, which is a circle.

12. Take one angle of the triangle as origin and a side as axis of x.



Let side of triangle $=a$; and let (x, y) be the co-ordinates of moving point P.

Now equation to AC is $y+\sqrt{3}x-a\sqrt{3}=0$,

$\therefore\ PE=\dfrac{y+\sqrt{3}x-a\sqrt{3}}{a}$, (see Art. 47)

And equation to

OA is $y-\sqrt{3}\,.\,x=0$;

$\therefore\ PF=\dfrac{y-\sqrt{3}\,.\,x}{2}$;

$$\therefore\ y^2+\left(\frac{y+\sqrt{3}x-a\sqrt{3}}{2}\right)^2+\left(\frac{y-\sqrt{3}x}{2}\right)^2=\text{constant}=c^2;$$

this reduces to $x^2+y^2-ax-\dfrac{ay}{\sqrt{3}}=\dfrac{2c^2}{3}-\dfrac{a^2}{2}$, which is a circle.

13. Let the fixed points be (h_1, k_1), (h_2, k_2), &c...., and (x, y) the moving point.

Hence we have $(x-h_1)^2+(y-k_1)^2+(x-h_2)^2+(y-k_2)^2+\&c.=\text{constant}$.

In this equation the coefficients of x^2 and y^2 are equal, hence it is a circle.

14. Let the equation to the circle be

$$x^2+y^2+2xy\cos\omega+Ax+By+C=0.$$

In this particular case, since the origin is on the circumference, $C=0$.

Also $\omega=120^0$, so that the equation becomes

$$x^2+y^2-xy+Ax+By=0.$$

This is to go through the point $(h, 0)$,

$$\therefore\ h^2+Ah=0;\ \therefore\ A=-h.$$

Similarly $\qquad B=-k,$

therefore the equation is $x^2+y^2-xy-hx-ky=0$.

15. Comparing the given equation with the form in Art. 104, we see that

$$2\cos\omega=-1,\ \therefore\ \omega=120^0.$$

Also $\qquad 2(a+b\cos\omega)=h$, and $2(b+a\cos\omega)=h$.

Hence $\qquad a=b=h,$

and $\qquad a^2+b^2+2ab\cos\omega-c^2=0,\ \therefore\ c=h.$

16. Referring to Art. 104, we see that $2\cos\omega=1$,

$$\therefore\ \omega=60^0.$$

Also $\qquad 2(a+b\cos\omega)=h,$

and $\qquad 2(b+a\cos\omega)=h,$

and $\qquad a^2+b^2+2ab\cos\omega-c^2=0.$

From these we obtain $\quad a=\dfrac{h}{3},\ b=\dfrac{h}{3},\ c=\dfrac{h}{\sqrt{3}}.$

17. Inserting the values $a=0$, $b=0$, $c=3$, $\omega=45^0$, in the general equation of Art. 104, we get $\quad x^2+y^2+\sqrt{2}\,.\,xy-9=0.$

18. Inserting the values $a=-\frac{1}{3}$, $b=-\frac{1}{3}$, $c=\frac{2}{\sqrt{3}}$; $\omega=60^0$, in the general equation of Art. 104, we get

$$x^2+y^2+xy+x+y-1=0.$$

19. Referring to Art. 104, we see that

$$(a+b\cos\omega)=\tfrac{1}{2}h,$$

and

$$(b+a\cos\omega)=\tfrac{1}{2}k,$$

and

$$a^2+b^2+2ab\cos\omega-c^2=0.$$

Hence we get $\qquad a=\dfrac{h-k\cos\omega}{2\sin^2\omega}$; $b=\dfrac{k-h\cos\omega}{2\sin^2\omega}$;

$$\therefore\ c=\frac{\sqrt{(h^2+k^2-2hk\cos\omega)}}{2\sin\omega}.$$

20. If c be the radius, it is easily seen that each co-ordinate of the centre is $\dfrac{c}{\sin\omega}$.

Hence, by Art. 104, the equation is

$$x^2+y^2-2cx\left(\frac{1+\cos\omega}{\sin\omega}\right)-2cy\left(\frac{1+\cos\omega}{\sin\omega}\right)+2xy\cos\omega+2c^2\left(\frac{1+\cos\omega}{\sin^2\omega}\right)-c^2=0,$$

or

$$x^2+y^2+2xy\cos\omega-2c(x+y)\cot\frac{\omega}{2}+c^2\cot^2\frac{\omega}{2}=0.$$

21. Since $\cos\omega=1-2\sin^2\frac{\omega}{2}$, the equation becomes

$$x^2+y^2+2xy-2c(x+y)\cot\frac{\omega}{2}+c^2\,.\,\cot^2\frac{\omega}{2}=4xy\sin^2\frac{\omega}{2}.$$

Take the square root and we have

$$x+y-c\,.\,\cot\frac{\omega}{2}=2\sqrt{xy}\,.\,\sin\frac{\omega}{2}.$$

22. In Example 10 the length of the common chord is found to be $\sqrt{4c^2-2(a-b)^2}$; if the circles touch, this is zero, and therefore $c=\dfrac{a\sim b}{\sqrt{2}}$.

23. Take a as the length of a side. Then the co-ordinates of centre are $\dfrac{a}{2}$ and $\dfrac{a}{2\sqrt{3}}$, and the radius is $\dfrac{a}{\sqrt{3}}$.

Hence the equation is

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{a}{2\sqrt{3}}\right)^2=\frac{a^2}{3},\ \text{or}\ x^2+y^2-ax-\frac{ay}{\sqrt{3}}=0.$$

This equation might also have been obtained by the method used in Ex. 3 (*Aliter*).

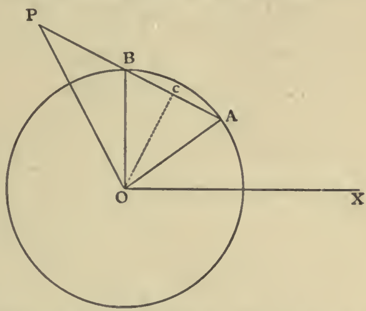
Substituting $r\cos\theta$, $r\sin\theta$ for x and y, we get

$$r^2 = ar\left(\cos\theta + \frac{1}{\sqrt{3}}\sin\theta\right),$$

or

$$r = \frac{2a}{\sqrt{3}} \cdot \cos(\theta - 30^0).$$

24. Let AB be the chord, so that $AOB = 2\beta$; $COx = \alpha$.



Take *any* point P on AB, and let its co-ordinates be (r, θ).

Now $$OP\cos POC = OC = OA \cdot \cos COA;$$

$$\therefore\ r \cdot \cos(\theta - \alpha) = c \cdot \cos\beta,$$

or

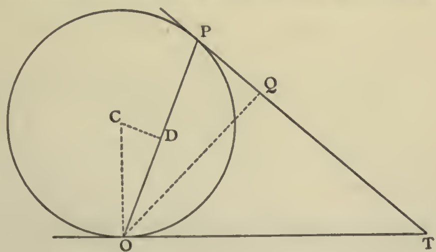
$$r = c \cdot \cos\beta \cdot \sec(\theta - \alpha).$$

As AB approaches the position of a tangent, the angle β diminishes and becomes 0 when AB is a tangent.

Hence the polar equation to the tangent required is

$$r = c \cdot \sec(\theta - \alpha).$$

25. Let P be *any* point on the circle, (r, θ') its co-ordinates.



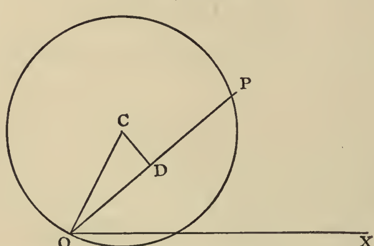
Now $$2OD = 2OC \cos COD,$$
$$\therefore\ r = 2c \,.\, \sin\theta',$$
which is the required equation to the circle, if we take θ' as a variable co-ordinate.

Secondly, take (r', θ') as the co-ordinates of the particular point at which the tangent is drawn. Let Q be *any* point on the tangent, and let its co-ordinates be (r, θ). Now

$$\angle OQT = POQ + OPQ = POQ + POT = \theta' - \theta + \theta' = 2\theta' - \theta.$$

Then $$\frac{OQ}{OP} = \frac{\sin OPQ}{\sin OQP} = \frac{\sin POT}{\sin OQT};$$
$$\therefore\ \frac{r}{2c \,.\, \sin\theta'} = \frac{\sin\theta'}{\sin(2\theta' - \theta)};$$
$$\therefore\ r\sin(2\theta' - \theta) = 2c \,.\, \sin^2\theta'.$$

26. Here $$OC = c,\quad COx = \alpha.$$



Let (r, θ) be co-ordinates of *any* point P.

Then $$OP = 2OD = 2OC \,.\, \cos COP,$$
or $$r = 2c \,.\, \cos(\theta - \alpha).$$

27. The equation will transform into the shape
$$r = M\cos\theta + N\sin\theta,$$
where M and N are constants.

Now let $M = k\cos A$, and $N = k\sin A$; then our equation becomes
$$r = k\cos(\theta - A),$$
which by the last example is the equation to a circle, the origin being on the circumference, and the diameter through the origin making an angle A with the initial line; the length of the diameter is k.

28. Take A as origin, and AB as initial line.

In the first case let the lines AL and AM make angles of β and $-\beta$ respectively with AB.

Let the diameter of the indefinite circle be $2c$, and let it make an angle α with AB, so that its equation is $r = 2c \,.\, \cos(\theta - \alpha)$, by Ex. 26.

Hence $\quad AL=2c\,.\cos(\beta-\alpha)$, and $AM=2c\,.\cos(-\beta-\alpha)$;

$$\begin{aligned}\therefore\quad AL+AM&=2c\,.\{\cos(\beta-\alpha)+\cos(-\beta-\alpha)\}\\&=4c\,.\cos\beta\,.\cos\alpha\\&=2AB\,.\cos\beta,\ \text{ since } AB=2c\,.\cos\alpha.\end{aligned}$$

Hence the sum of AL and AM is independent of c and α.

Secondly, let the lines AL, AM, make angles β and $180^0-\beta$ with AB.

Then $\quad AL=2c\,.\cos(\beta-\alpha)$,

and $\quad AM=2c\,.\cos(180-\beta-\alpha)$;

$$\begin{aligned}\therefore\quad AL-AM&=2c\,.\{\cos(\beta-\alpha)-\cos(180-\beta-\alpha)\}\\&=2c\,.\{\cos(\beta-\alpha)+\cos(\beta+\alpha)\}\\&=4c\,.\cos\beta\,.\cos\alpha\\&=2AB\,.\cos\beta,\ \text{as before.}\end{aligned}$$

29. Take AC as initial line, and a as length of side of triangle. Let $PA=r$, $PAC=\theta$; then $PA=PC+PB$, or

$$r=\sqrt{r^2+a^2-2ar\cos\theta}+\sqrt{r^2+a^2-2ar\,.\cos(60^0-\theta)}.$$

Transpose one term to the left-hand side, and square and reduce; we get

$$r-2a\{\cos\theta-\cos(60^0-\theta)\}=2\sqrt{r^2+a^2-2ar\cos\theta}.$$

Squaring again and simplifying, we get

$$3r^2-4\sqrt{3}\,.\,ar\,.\cos(30^0-\theta)+4a^2\,.\cos^2(30^0-\theta)=0.$$

Taking the square root, we get

$$\sqrt{3}\,.\,r-2a\,.\cos(30^0-\theta)=0,\quad\text{or}\quad r=\frac{2a}{\sqrt{3}}\,.\cos(\theta-30^0).$$

By Ex. 26 this is the equation to a circle with the origin on the circumference, and the diameter through the origin making an angle of 30^0 with the initial line; hence the diameter through A bisects angle BAC, and its length is $\frac{2a}{\sqrt{3}}$; hence it is easily seen that the circle is the circumscribing circle of ABC.

Or, we might at once make use of the result of Ex. 23, and thus see that the circle found was the circle ABC.

Or again, in the equation $r=\frac{2a}{\sqrt{3}}\,.\cos(\theta-30^0)$ put $\theta=0$, therefore $r=a$. Hence the circle goes through C. Also, put $\theta=60^0$, therefore $r=a$. Hence the circle goes through B. Consequently it is the circle ABC.

[*Note.* The circle is divided into three arcs by the sides of the triangle; one of these arcs is the locus of P when

$$PA=PB+PC;$$

another the locus when $\quad PB=PA+PC$;

and the other the locus when $\quad PC=PA+PB$.]

30. Take the fixed straight line as axis of x, and a line perpendicular to it as axis of y.

Let co-ordinates of P be (h, k), and let one of the given straight lines be

$$y - x \tan \alpha - c_1 = 0.$$

The square of perpendicular on this from P is

$$\frac{(k - h \tan \alpha - c_1)^2}{1 + \tan^2 \alpha}, \text{ which } = (k \cos \alpha - h \sin \alpha - c_1 \cos \alpha)^2.$$

Hence the equation to the locus of P is

$$(k \cos \alpha - h \sin \alpha - c_1 \cos \alpha)^2 + (k \cos \beta - h \sin \beta - c_2 \cos \beta)^2 + \&c.$$
$$= \text{constant} = m^2 \text{ suppose :}$$

this may be written

$$k^2 (\cos^2 \alpha + \cos^2 \beta + \ldots\ldots) + h^2 (\sin^2 \alpha + \sin^2 \beta + \ldots\ldots)$$
$$- 2hk (\sin \alpha \,.\, \cos \alpha + \sin \beta \,.\, \cos \beta + \ldots\ldots) + Ah + Bk + C = 0,$$

where ABC are constants.

If this is the equation to a circle, the coefficient of h^2 must equal that of k^2, and coefficient of hk is zero.

Hence we have $\cos^2 \alpha + \cos^2 \beta + \ldots\ldots = \sin^2 \alpha + \sin^2 \beta + \ldots\ldots$

and $$2 \sin \alpha \,.\, \cos \alpha + 2 \sin \beta \,.\, \cos \beta + \ldots\ldots = 0.$$

These may be written

$$\cos 2\alpha + \cos 2\beta + \ldots\ldots = 0,$$

and $$\sin 2\alpha + \sin 2\beta + \ldots\ldots = 0.$$

31. Let the polygon have n sides; then (by Eucl. I. 32, Cor.) all its exterior angles are equal to 2π, and therefore each exterior angle is $\frac{2\pi}{n}$.

Take one side as axis of x, then the other sides make angles of

$$\frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n} \ldots\ldots \text{ with it.}$$

Hence the conditions in the previous example become

$$\sin 0^0 + \sin \frac{4\pi}{n} + \sin \frac{8\pi}{n} + \ldots\ldots \text{ to } n \text{ terms} = 0,$$

and $$\cos 0^0 + \cos \frac{4\pi}{n} + \cos \frac{8\pi}{n} + \ldots\ldots \text{ to } n \text{ terms} = 0.$$

But, by the methods in Todhunter's *Trigonometry*, Chap. XXII., it is easily seen that these conditions are satisfied, so that the locus is a circle.

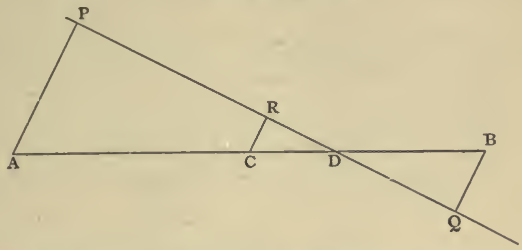
32. Let PQ make an angle α with AB.

And, *first*, let AP and BQ be on opposite sides of PQ.

Then $\qquad AP = AD \,.\, \sin\alpha,$

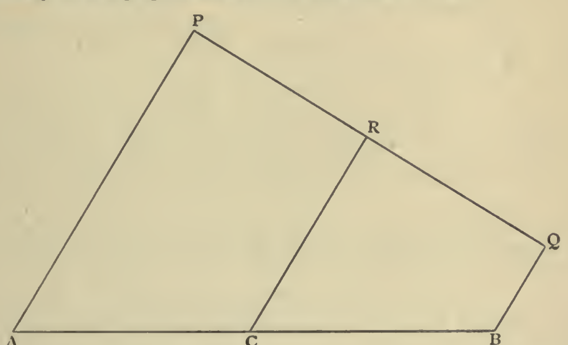
and $\qquad BQ = DB \,.\, \sin\alpha;$

$$\therefore\quad AP + BQ = AB \sin\alpha.$$



Hence $AB \,.\, \sin\alpha$ is to be constant, and therefore the angle α is constant; hence the angle RCD is constant, or in other words the locus of R is a straight line through C making a fixed angle with AB.

Secondly, let the perpendiculars lie on the same side of PQ.



Then it is easily seen that

$$CR = \tfrac{1}{2}(AP + BQ),$$

and as $AP + BQ$ is constant, it follows that CR is constant.

Hence R is at a constant distance from C, and consequently its locus is a circle.

[*Note*. In the first case it is evident that PQ might make an angle α on the other side of AB, so that the locus of R will consist of *two* straight lines through C equally inclined to AB in opposite directions.]

33. Let OD be drawn perpendicular to AB, and let its length be a. Let the angle $DOP=\theta$, and $OQ=r$.

Now $$OP=a\sec\theta;$$

$$\therefore\quad ra\sec\theta=k^2,$$

or $$r=\frac{k^2}{a}\cdot\cos\theta,$$

which is the polar equation to a circle.

34. Let C be the centre of the circle, and let

$$OC=a,\quad OQ=r,\quad COQ=\theta,\quad CP=c.$$

Now $$CP^2=OP^2+OC^2-2OP\,.\,OC\,.\cos\theta$$

$$=\frac{k^4}{OQ^2}+OC^2-\frac{2k^2}{OQ}\,.\,OC\,.\cos\theta;$$

$$\therefore\quad c^2=\frac{k^4}{r^2}+a^2-\frac{2k^2a\cos\theta}{r},$$

or $$r^2+2r\cos\theta\,\frac{ak^2}{c^2-a^2}-\frac{k^4}{c^2-a^2}=0.$$

By Art. 105 we see that this is a circle.

35. The equation to a tangent is $y=mx+c\sqrt{1+m^2}$. If this goes through the point (h, k) we shall have

$$k=mh+c\sqrt{1+m^2}.$$

Now, *theoretically*, we should solve this equation to find m; it would be a quadratic, and would therefore have two roots, and each of these if inserted in the first equation would give the equation to a tangent through (h, k). These two tangents would then be combined into one equation in the usual way.

But, *practically*, this mode of operation would be clumsy and laborious, and can be shortened thus: it is evident that our required equation will result from the elimination of m between the two equations

$$y=mx+c\sqrt{1+m^2},\quad\text{and}\quad k=mh+c\sqrt{1+m^2},$$

so that we may perform this elimination in whatever way is most convenient.

Subtracting one equation from the other, we get

$$y-k=m(x-h),\quad\text{or}\quad m=\frac{y-k}{x-h}.$$

Substitute this in the first of the two equations, and we get

$$y=x\,.\,\frac{y-k}{x-h}+c\sqrt{1+\frac{(y-k)^2}{(x-h)^2}}.$$

If this is cleared of fractions, we get

$$(kx-hy)^2=c^2\{(x-h)^2+(y-k)^2\}.$$

35. (*Aliter.*) The equation

$$(x^2+y^2-c^2)(h^2+k^2-c^2)=(hx+ky-c^2)^2$$

is evidently satisfied when $x=h$ and $y=k$; hence it is some locus going through (h, k).

Also it is satisfied when the equations $x^2+y^2-c^2=0$ and $xh+yk-c^2=0$ are *simultaneously* true; hence it is some locus passing through the points of contact of the tangents from (h, k). (See Art. 100.)

Also, by a method similar to that in Ex. 38, Chap. IV., it may be shewn that the equation does represent two straight lines; consequently it does represent the two tangents.

This equation can be readily transformed into the shape given in the question.

36. The given equation can be written

$$r^2-ar(2\cos\theta-\sec\theta)-2a^2=0,$$

or

$$(r-2a\cos\theta)(r+a\sec\theta)=0.$$

Hence it represents the circle $r=2a\cos\theta$ and the straight line

$$r=-a\sec\theta.$$

37. By Art. 28 it is evident that the equation

$$2c\,.\cos\beta\,.\cos\alpha=r\cos(\beta+\alpha-\theta)$$

is the equation to *some* straight line; solving it simultaneously with the equation to the circle, we get for the points of intersection

$$2\cos\beta\,.\cos\alpha=2\cos\theta\,.\cos(\beta+\alpha-\theta),$$

or

$$\cos(\alpha+\beta)+\cos(\alpha-\beta)=\cos(\alpha+\beta)+\cos(\alpha+\beta-2\theta),$$

or

$$\cos(\alpha+\beta-2\theta)=\cos(\alpha-\beta).$$

But since the origin is on the circumference and the initial line a diameter, it is evident that at the points of intersection θ cannot exceed $\frac{\pi}{2}$; hence our last equation will give

$$\alpha+\beta-2\theta=\pm(\alpha-\beta),$$

from which it follows that $\theta=\alpha$ or β.

[*Note.* By putting $\beta=\alpha$, we get $2c\cos^2\alpha=r\cos(2\alpha-\theta)$ as the tangent at the point α.]

38. Let the centre of the circle be C, and O be the extremity of the first-named diameter; take O as origin and OC as initial line. Let

$$POC=\alpha,\quad QOC=\beta.$$

Then, by the Note to the last example, the equation to tangent PT is

$$2c\,.\cos^2\alpha=r\,.\cos(2\alpha-\theta),$$

and the tangent QT is $\quad 2c\,.\cos^2\beta=r\,.\cos(2\beta-\theta).$

Hence at the point T we have

$$\frac{\cos(2\alpha-\theta)}{\cos^2\alpha}=\frac{\cos(2\beta-\theta)}{\cos^2\beta}, \text{ where } \theta=TOC;$$

whence we get $$2\tan\theta=\tan\alpha+\tan\beta,$$

or $$\frac{2Ct}{OC}=\frac{Cp}{OC}+\frac{Cq}{OC};$$

$$\therefore\ 2\,Ct=Cp+Cq;$$

$$\therefore\ Ct-Cq=Cp-Ct,$$

$$\therefore\ tq=pt.$$

39. Let the given points be (x_1, y_1) and (x_2, y_2) and (x_3, y_3).

Then, if we represent our circle by the equation

$$x^2+y^2+Ax+By+C=0,$$

we must have $$x_1^2+y_1^2+Ax_1+By_1+C=0,$$

and $$x_2^2+y_2^2+Ax_2+By_2+C=0,$$

and $$x_3^2+y_3^2+Ax_3+By_3+C=0.$$

From these three equations we obtain the values of A, B, C, and can then insert them in the original equation.

If Δ be the area of the triangle formed by joining the three given points (see Art. 11), then we have

$$A=\frac{x_1^2(y_2-y_3)+x_2^2(y_3-y_1)+x_3^2(y_1-y_2)+y_1^2(y_2-y_3)+y_2^2(y_3-y_1)+y_3^2(y_1-y_2)}{2\Delta},$$

and similarly for B and C.

40. The co-ordinates of the centre and the radius of the circle are (by Art. 88) simple functions of the quantities A, B, C determined in the last example; hence they will remain finite as long as A, B, C remain finite. But A, B, C only become infinite when their denominators are zero, that is when $\Delta=0$. But in this case the three points are in one straight line (see Art. 36).

CHAPTER VII.

1. Writing the equations to the lines in the shapes

$$y=-\frac{b}{a}x+b \text{ and } y=-\frac{b'}{a'}x+b',$$

the required tangent $$=\frac{\dfrac{b'}{a'}-\dfrac{b}{a}}{1+\dfrac{bb'}{aa'}}=\frac{ab'-a'b}{aa'+bb'}.$$

2. The equation to two straight lines through the origin making angles α, β with the axis is

$$(y - x \cdot \tan\alpha)(y - x \cdot \tan\beta) = 0;$$

or

$$y^2 - xy(\tan\alpha + \tan\beta) + x^2 \tan\alpha \cdot \tan\beta = 0.$$

But this equation is to be identical with the given equation, which may be written

$$y^2 - 2xy \tan\phi \cdot \operatorname{cosec}^2\phi + x^2(\tan^2\phi + \cos^2\phi)\operatorname{cosec}^2\phi = 0.$$

Hence (Introd. § VIII.)

$$\begin{aligned}\tan\alpha + \tan\beta &= 2\tan\phi \cdot \operatorname{cosec}^2\phi\\ &= 2\operatorname{cosec}\phi \cdot \sec\phi,\end{aligned}$$

and

$$\begin{aligned}\tan\alpha \cdot \tan\beta &= (\tan^2\phi + \cos^2\phi)\operatorname{cosec}^2\phi\\ &= \sec^2\phi + \cot^2\phi;\end{aligned}$$

therefore (Introd. § v.)

$$\begin{aligned}(\tan\alpha \sim \tan\beta)^2 &= 4\operatorname{cosec}^2\phi \cdot \sec^2\phi - 4(\sec^2\phi + \cot^2\phi)\\ &= 4\sec^2\phi(\operatorname{cosec}^2\phi - 1) - 4\cot^2\phi\\ &= 4\sec^2\phi \cdot \cot^2\phi - 4\cot^2\phi\\ &= 4\cot^2\phi(\sec^2\phi - 1)\\ &= 4\cot^2\phi \cdot \tan^2\phi = 4;\end{aligned}$$

$$\therefore \quad \tan\alpha \sim \tan\beta = 2.$$

3. Let the square be $OABC$, and let OA make an angle α with the axis Ox. Let a be length of OA. Then equation to OA is $y = x\tan\alpha$: to OB it is $y = x\tan(\alpha + 45)$; to OC is $y = x\tan(\alpha + 90)$ or $y = -x\cot\alpha$: to AB is $y = -x\cot\alpha + a\operatorname{cosec}\alpha$; to AC is $y = -x \cdot \tan(45^0 - \alpha) + a\sin 45^0 \cdot \sec(45^0 - \alpha)$; and to BC is $y = x \cdot \tan\alpha + a\sec\alpha$.

4. The equation $\frac{x}{a} + \frac{y}{b} = \frac{x}{b} + \frac{y}{a}$ is evidently satisfied when the first and third equations are *simultaneously* true, and also when the second and fourth are *simultaneously* true; hence it is the equation to one diagonal. It may be written in the shape $y = x$.

Again, the equation $\frac{x}{a} + \frac{y}{b} + \frac{x}{b} + \frac{y}{a} = 1 + 2$ is evidently satisfied when the first and fourth equations are *simultaneously* true, and also when the second and third are *simultaneously* true; hence it is the equation to the other diagonal. It may be written $y = -x + \frac{3ab}{a+b}$, and it is therefore a straight line at right angles to the other diagonal.

5. Let $y = mx$ be the equation to a straight line through the origin; the perpendicular on it from (x_1, y_1) is

$$\frac{y_1 - mx_1}{\sqrt{1 + m^2}};$$

Hence $$\frac{y_1 - mx_1}{\sqrt{1+m^2}} = \delta.$$

Eliminating m between this and the equation $y = mx$, we get the required result. [See remarks upon Ex. 35 of Chapter VI.]

6. It is evidently necessary that one of the factors of $Ay^2 + Bxy + Cx^2$ should be a factor of $ay^2 + bxy + cx^2$, or in other words that it should be a common measure of these two expressions. Hence, perform the operation for finding their G.C.M., and equate the final remainder to zero, as there cannot be any such remainder if they have a common factor.

We get the following condition:

$$A^2c^2 + C^2a^2 + B^2ac + ACb^2 - 2ACac - ABbc - abBC = 0.$$

7. Take the outer triangle as the triangle of reference, and r the radius of the inscribed circle of this triangle; then the co-ordinates of the inscribed centre are $\alpha = r$, $\beta = r$, $\gamma = r$, and these co-ordinates evidently satisfy the given equation.

Again, let r' be the inscribed radius of the inner triangle, then the co-ordinates of the inscribed centre of this triangle are $\alpha = r' + a$, $\beta = r' + b$, $\gamma = r' + c$, and these evidently satisfy the given equation.

8. Let E be the middle point of BC, then its co-ordinates are

$$\left(0,\ \frac{a}{2}\sin C,\ \frac{a}{2}\sin B\right).$$

Now the equation to the external bisector at A is $\beta + \gamma = 0$ [Art. 72], hence the equation to the line through E is

$$\beta + \gamma = \text{constant} = k.$$

But this equation is to be satisfied by the co-ordinates of E;

$$\therefore\ \frac{a}{2}\sin C + \frac{a}{2}\sin B = k.$$

Hence the required equation is

$$\beta + \gamma = \frac{a}{2}(\sin C + \sin B).$$

9. The centre of the escribed circle touching BC is the point of intersection of $\alpha + \beta = 0$ with $\alpha + \gamma = 0$, therefore any straight line through this point must be of the form

$$l(\alpha + \beta) + m(\alpha + \gamma) = 0.$$

But since this line is to be parallel with the line $\alpha = 0$, it must be of the form $\alpha + \text{constant} = 0$, or $\alpha + k(a\alpha + b\beta + c\gamma) = 0$ (see Art. 73).

Comparing the coefficients of β and γ in the two forms, we get

$$l : m :: b : c :: \sin B : \sin C.$$

Hence the equation is

$$(\alpha + \beta)\sin B + (\alpha + \gamma)\sin C = 0.$$

10. If the sines are in a given ratio, it follows that the perpendiculars from any point in either dividing line are in the same ratio. Let this ratio be $q : p$.

Hence if (α, β, γ) be any point on either dividing line it follows, by Art. 5, Chap. IV., that

$$\frac{l\alpha+m\beta+n\gamma}{\sqrt{(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)}}$$
$$: \pm\frac{l'\alpha+m'\beta+n'\gamma}{\sqrt{(l'^2+m'^2+n'^2-2m'n'\cos A-2n'l'\cos B-2l'm'\cos C)}} :: q : p.$$

The positive sign gives the equation to one dividing line, the negative the other.

11. Let $y=m_1x$ and $y=m_2x$ be the two straight lines represented by

$$Ay^2+Bxy+Cx^2=0,$$

so that
$$m_1+m_2=-\frac{B}{A}, \text{ and } m_1m_2=\frac{C}{A}.$$

Let $y=px$ be one of the bisectors, then since it makes equal angles with the two straight lines $y=m_1x$ and $y=m_2x$,

we have
$$\frac{p-m_1}{1+pm_1}=\frac{m_2-p}{1+pm_2},$$

or
$$p^2(m_1+m_2)+2p(1-m_1m_2)-(m_1+m_2)=0,$$

or
$$Bp^2-B=2p(A-C).$$

Eliminating p between this and the equation $y=px$, we get

$$By^2-Bx^2=2xy(A-C).$$

[See remarks on Ex. 35, Chap. VI.]

12. By the previous question the two sets of bisectors are

$$\frac{2xy}{y^2-x^2}=\frac{B}{A-C},$$

and
$$\frac{2xy}{y^2-x^2}=\frac{b}{a-c}.$$

If these are identical, it is obvious that

$$\frac{B}{A-C}=\frac{b}{a-c}.$$

13. Since the equation $u+\lambda v=0$ is satisfied when $u=0$ and $v=0$ are *simultaneously* true, it evidently represents some locus through the intersections of the two circles.

Also, by transforming to rectangular co-ordinates, it is easily seen that the coefficients of x^2 and y^2 in the equation $u+\lambda v=0$ will be equal; hence the equation denotes a circle. And it may be of any size by properly choosing the value of λ.

14. Let the fixed circle be $u=0$. Also let $v=0$ and $w=0$ be two *fixed* circles each passing through the two fixed points.

Every other circle through these two points can be represented by

$$v+lw=0.$$

Let the equations $u=0$, $v=0$, $w=0$, be expressed in their simplest form in rectangular co-ordinates,—that is to say, let the coefficients of x^2 and y^2 in each equation be unity. Then the equation $(1+l)\,u-(v+lw)=0$ will not contain x^2 or y^2 and is therefore the equation to a straight line; also it is satisfied when $u=0$ and $v+lw=0$ are *simultaneously* true, and is therefore the common chord of these two circles.

Now writing this equation in the form $u-v+l\,(u-w)=0$, it is evident that whatever be the value of l, this chord passes through the intersection of the fixed straight line $u-v=0$ with the fixed straight line $u-w=0$; that is to say, it passes through a fixed point.

CHAPTER VIII.

1. The equation to AL is

$$y=\frac{2a}{a}x, \text{ or } y=2x.$$

2. The centre O will be on the axis. Let $AO=h$.

$$\therefore\ OL^2=OS^2+SL^2=(h-a)^2+4a^2.$$

But $OL=h$; since AO and OL are both radii.

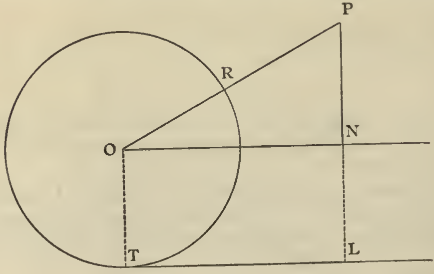
$$\therefore\ h^2=(h-a)^2+4a^2;$$

$$\therefore\ h=\frac{5a}{2}.$$

Hence the required circle is $\left(x-\frac{5a}{a}\right)^2+y^2=\left(\frac{5a}{2}\right)^2$,

or
$$x^2+y^2-5ax=0.$$

3. Let ON be the fixed diameter; draw TL a tangent parallel to ON.



Now, by hypothesis, $PR = PN$;

$$\therefore\ PO = PL.$$

Hence the distance of P from O is equal to its distance from the fixed straight line TL; therefore the locus of P is a parabola whose focus is O and directrix TL.

3. (*Aliter.*) Taking ON as axis of x (see preceding figure) let the co-ordinates of P be (x, y) and let the radius of circle $= c$.

Then
$$PR = PN;$$
$$\therefore\ OP - c = y,$$
$$\therefore\ OP^2 = (c+y)^2,$$
$$\therefore\ x^2 + y^2 = c^2 + 2cy + y^2,$$
or
$$x^2 = c^2 + 2cy = 2c\left(y + \frac{c}{2}\right).$$

If the origin was transferred to the point $\left(0,\ -\frac{c}{2}\right)$ the equation would become $x^2 = 2cy$, that is to say it would be of the form it assumes when referred to vertex as origin. Hence the vertex bisects the line OT. Also, from the form of the equation we see that the latus rectum is $2c$, and therefore the distance of the vertex from either focus or directrix is $\frac{c}{2}$. Hence O is focus, and TL the directrix. (See also the next example.)

4. The first curve is the parabola in its usual position with the axis of x for its axis.

The second curve can be written $x^2 = -4ay$; hence we see that it bears the same relative position to the axis of y that the other did to the axis of x. Also it is evident that only negative values of y will give real values of x; hence it has the negative part of the axis of y as its axis.

Solving the equations simultaneously we find that the curves cut at the origin and at the point $(4a,\ -4a)$.

5. In the general form of the equation to a tangent, namely

$$yy' = 2a(x + x')$$

insert the values $x' = a$ and $y' = 2a$, and the equation becomes $y = x + a$.

6. The two lines are $y = 2x$ and $y = x + a$; hence tangent of included angle $= \frac{2-1}{1+2\times 1} = \frac{1}{3}$.

7. By Art. 134, the normal is

$$y - 2a = -(x - a), \text{ or } y = -x + 3a.$$

8. Solving simultaneously the equations $y = -x + 3a$ and $y^2 = 4ax$, we get the equation $x^2 - 10ax + 9a^2 = 0$, the roots of which are a and $9a$. When $x = 9a$, $y = -6a$, so that the normal meets the curve again in the point $(9a,\ -6a)$.

The length of the intercepted chord is

$$\sqrt{(9a - a)^2 + (6a + 2a)^2}, \text{ or } 8a\sqrt{2}.$$

9. The equation to a tangent is $y=mx+\frac{a}{m}$, where $m=\frac{2a}{y'}$. In this case $\frac{2a}{y'}=\tan 30^\circ$.

$$\therefore\ y'=2a\sqrt{3};\ \text{whence we get}\ x'=3a.$$

10. The equation to the tangent at (x', y') is $yy'=2a(x+x')$; and by Art. 49, the perpendicular on this from $(-a, 0)$ is

$$\pm\frac{2a^2-2ax'}{\sqrt{\{y'^2+4a^2\}}},\ \text{or}\ \pm\frac{a(a-x')}{\sqrt{\{a(x'+a)\}}}\ \text{since}\ y'^2=4ax'.$$

11. Referring to the previous example, we have in the present case

$$\pm\frac{a(a-x')}{\sqrt{\{a(x'+a)\}}}=a.$$

Hence we get $x'=0$, or $3a$.

Hence the points of contact are $(0, 0)$, $(3a, 2a\sqrt{3})$, $(3a, -2a\sqrt{3})$.

12. The equation to the circle is $x^2+y^2=\left(\frac{3a}{2}\right)^2$; solving this with the equation to the parabola we get $x=\frac{a}{2}$, or $-\frac{9a}{2}$. The first of these two is the only one applicable; hence the common chord bisects AS.

13. The equation may be written $(x-\frac{1}{2})^2=\frac{1}{4}-y$; if now the origin be transferred to the point $(\frac{1}{2}, \frac{1}{4})$ the equation becomes $x^2=-y$, which is the shape of a parabola referred to vertex as origin with its axis coinciding with the negative part of the axis of y. Hence the curve is a parabola whose vertex is at the point $(\frac{1}{2}, \frac{1}{4})$, with its axis parallel to the axis of y, the branches extending to an infinite distance *downwards*.

Solving the two given equations simultaneously we get two coincident values of x namely 1, $\therefore\ y=0$: hence the line is a tangent at the point $(1, 0)$.

14. In the figure to Art. 125 let the tangent at P meet the latus rectum in R and the directrix in K.

Take equation to tangent as $y=mx+\frac{a}{m}$.

Put $$x=-a,\ \therefore\ OK=\frac{a}{m}-ma.$$

$$\therefore\ SK^2=SO^2+OK^2=4a^2+\left(\frac{a}{m}-ma\right)^2=\left(\frac{a}{m}+ma\right)^2.$$

$$\therefore\ SK=\frac{a}{m}+ma.$$

In equation to tangent put $x=a$,

$$\therefore\ SR=\frac{a}{m}+ma=SK.$$

15. Let QL be the ordinate of Q, so that $QL = \frac{1}{2} PM$.

Now, by equation to parabola,

$$4a \, . \, AL = QL^2 = \frac{PM^2}{4} = a \, . \, AM,$$

$$\therefore \; AL = \tfrac{1}{4} AM.$$

$$\therefore \; LM = \tfrac{3}{4} AM.$$

And, by similar triangles $AT : QL :: AM : LM$,

$$:: 4 : 3.$$

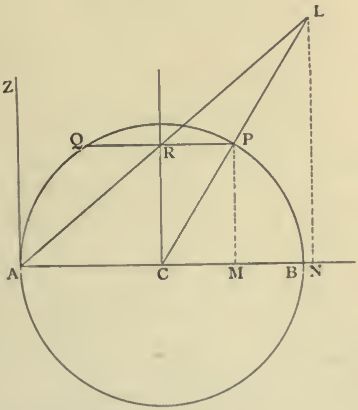
$$\therefore \; AT = \tfrac{4}{3} QL = \tfrac{2}{3} PM.$$

15. (*Aliter.*) Let the co-ordinates of P be (h, k): then, as in the previous proof, it is easily shewn that those of Q are $(\frac{1}{4} h, \frac{1}{2} k)$.

$\therefore$ equation to QM is $\qquad y = -\dfrac{2k}{3h}(x - h).$

Put $x = 0$ and we get $\qquad AT = \dfrac{2k}{3}.$

16. Now $\qquad \dfrac{LN}{AN} = \dfrac{RC}{AC} = \dfrac{PM}{CP} = \dfrac{LN}{CL},$



$\therefore \; CL = AN =$ perpendicular from L on AZ.

Hence locus of L is a parabola with C as focus and AZ as directrix.

16. (*Aliter.*) Take C as origin, and $CA=a$. Let (h, k) be co-ordinates of P; the equation to CP is $y=\frac{k}{h}x$.

The co-ordinates of R are $(0, k)$, and hence the equation to AR is

$$y=\frac{k}{a}(x+a).$$

Solving these two equations simultaneously, we get for the point L the equations
$$x=\frac{ah}{a-h},\ y=\frac{ak}{a-h}.$$

If now we eliminate h, k from these equations by the help of the condition $k^2+h^2=a^2$, we get $y^2=a^2+2ax$ as the locus of L. This is easily seen to be the parabola whose focus is C and directrix AZ.

17. Combine the equation $y=mx+c$ with the equation $y^2=4ax$, and we get $y^2-\frac{4a}{m}y+\frac{4ac}{m}=0$: this equation will determine the required ordinates, which we will call y_1 and y_2.

The ordinate of the middle point of the chord is $\frac{y_1+y_2}{2}$, which$=\frac{2a}{m}$: (Introduction § II.)

The values of y_1 and y_2 are as follows:

$$y_1=\frac{2a}{m}+\frac{\sqrt{(4a^2-4acm)}}{m},$$

$$y_2=\frac{2a}{m}-\frac{\sqrt{(4a^2-4acm)}}{m}.$$

18. Let PM be the ordinate of P; $AB=a$.

Now
$$\frac{PM}{AM}=\tan PAM=\cot BAQ=\frac{a}{BQ};$$

but $BQ=PM$, hence the equation becomes $PM^2=a\,.\,AM$, which is the equation to a parabola whose vertex is A, and latus rectum$=a$.

19. Let CP, AQ meet at L, and let PM, LN be the ordinates of P and L.

Let $y^2=4ax$ be the given parabola.

$$\therefore\ PM^2=4a\,.\,AM.$$

Now
$$\frac{AN}{LN}=\frac{AB}{QB}=\frac{AC}{PM};\ \text{and}\ \frac{CN}{LN}=\frac{CM}{PM}.$$

Hence, by subtraction,
$$\frac{AC}{LN}=\frac{AM}{PM};$$

$$\therefore\ \frac{AC}{LN}\cdot\frac{AN}{LN}=\frac{AM}{PM}\cdot\frac{AC}{PM}=\frac{AC}{4a};$$

$$\therefore\ LN^2=4a\,.\,AN,$$

and therefore L is on the parabola.

19. (*Aliter.*) Let the co-ordinates of Q be (h, k); now the ordinate of P being also k, it follows from the equation to the curve that its abscissa is $\frac{k^2}{4a}$. Also the co-ordinates of C are $(-h, 0)$.

Hence the equation to AQ is $y = \frac{k}{h} \cdot x$,

and equation to CP is $$y = \frac{4ak}{k^2+4ah}(x+h);$$
solving these simultaneously, we find the co-ordinates of L to be
$$\left(\frac{4ah^2}{k^2}, \frac{4ah}{k}\right);$$
and these satisfy the equation to the curve.

20. Solving the equations $y^2 = 4ax$ and $y - y' = -\frac{y'}{2a}(x - x')$ simultaneously, we get the quadratic $y^2 + \frac{8a^2}{y'} \cdot y - 8a^2 - y'^2 = 0$.

The roots of this equation are the ordinates of the points at which the normal cuts the curve; hence one root is y'. But the sum of the roots is (Introduction § II.) $-\frac{8a^2}{y'}$: hence the other root is $-\frac{8a^2}{y'} - y'$.

And the abscissa of this point is $\frac{1}{4a}\left(-\frac{8a^2}{y'} - y'\right)^2$,

or $$\frac{1}{4a}\left(\frac{8a^2}{y'} + y'\right)^2.$$

Also square of chord
$$= \left\{\frac{(8a^2+y'^2)^2}{4ay'^2} - x'\right\}^2 + \left\{\frac{8a^2}{y'} + 2y'\right\}^2,$$
$$= \left\{\frac{(8a^2+y'^2)^2}{4ay'^2} - \frac{y'^2}{4a}\right\}^2 + \left\{\frac{8a^2+2y'^2}{y'}\right\}^2,$$
$$= \left\{\frac{4a\,(4a^2+y'^2)}{y'^2}\right\}^2 + \left\{\frac{2\,(4a^2+y'^2)}{y'}\right\}^2,$$
$$= \frac{4}{y'^4}(4a^2+y'^2)^3.$$
$$\therefore \text{ length of chord } = \frac{2}{y'^2}(4a^2+y'^2)^{\frac{3}{2}}.$$

21. The result of the last example can be written
$$\frac{2}{4ax'}(4a^2+4ax')^{\frac{3}{2}}.$$

Now, by Art. 129, $$r = x' + a.$$
$$\therefore PQ = \frac{1}{2a(r-a)}(4ar)^{\frac{3}{2}} = \frac{8a^{\frac{3}{2}}r^{\frac{3}{2}}}{2a(r-a)} = \frac{4r\sqrt{ar}}{r-a}.$$

But, by figure to Art. 136, it is evident that

$$SZ^2 = AS \,.\, ST = AS \,.\, SP : \text{(Euclid VI. 8 Cor.)},$$

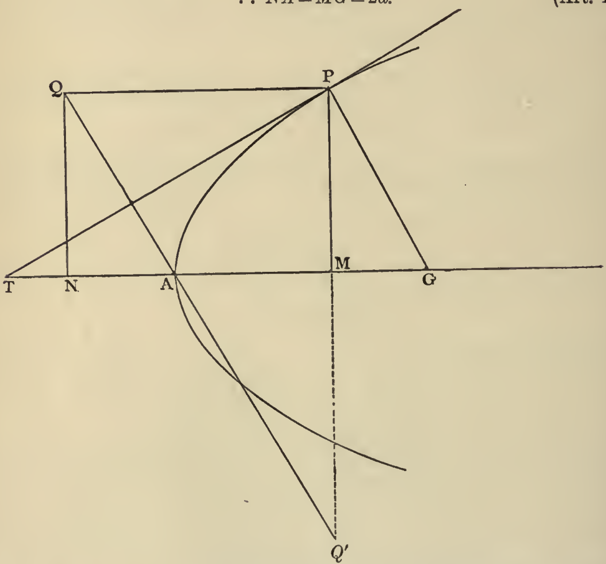
$$\therefore\ p^2 = ar.$$

$$\therefore\ PQ = \frac{4rp}{r-a}.$$

22. Draw the normal PG.

Then by similar triangles $QN : NA :: PM : MG$;

$$\therefore\ NA = MG = 2a. \qquad \text{(Art. 137.)}$$



Hence the locus of Q is a straight line parallel to the axis of y and at a distance $2a$ to the left of it.

Now let co-ordinates of P be (h, k), and of Q' be (h, l).

Then by similar triangles $Q'AM$ and QAN, we have

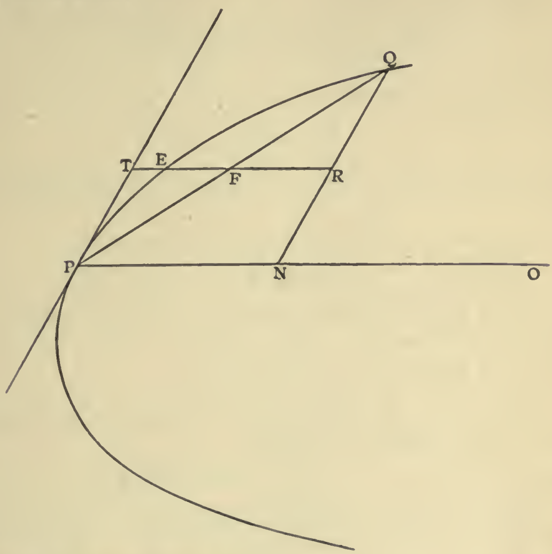
$$\frac{l^2}{h^2} = \frac{k^2}{4a^2} = \frac{4ah}{4a^2} = \frac{h}{a};$$

$$\therefore\ h^3 = al^2.$$

Hence Q' is a point on the curve $x^3 = ay^2$.

23. Take PN and PT as axes.

Let QN be ordinate of Q.



Take (x_1, y_1) as ordinates of E,
and (x_2, y_2) as ordinates of Q.

Equation to PQ is $y=\frac{y_2}{x_2}.x$;
and equation to TF is $y=y_1$.

Hence abscissa of F is $\frac{y_1x_2}{y_2}$.

$$\therefore\ EF=TF-TE=\frac{y_1x_2}{y_2}-x_1=\frac{y_1x_2-x_1y_2}{y_2};$$

$$\therefore\ \frac{TE}{EF}=\frac{x_1y_2}{x_2y_1-x_1y_2}=\frac{4ax_1y_2}{4ax_2y_1-4ax_1y_2}=\frac{y_2y_1^2}{y_1y_2^2-y_2y_1^2}=\frac{y_1}{y_2-y_1}.$$

But $$RN=y_1 \text{ and } RQ=y_2-y_1;$$

$$\therefore\ \frac{TE}{EF}=\frac{RN}{RQ}=\frac{PF}{FQ}.$$

24. *First,* suppose both points on the same side of the axis; let (h, k) be the co-ordinates of one point, and $(\mu h, \sqrt{\mu}k)$ those of the other.

The tangents at these points are

$$y = \frac{2a}{k}x + \frac{k}{2} \text{ and } y = \frac{2a}{\sqrt{\mu}\,.\,k}x + \frac{\sqrt{\mu}\,.\,k}{2}.$$

Hence for the point of intersection $x = \sqrt{\mu}\,.\,h$,

$$\text{and } y = \frac{k}{2}(1 + \sqrt{\mu}).$$

Eliminate (h, k) from these equations by help of the condition $k^2 = 4ah$, and we get

$$y^2 = ax\left(\sqrt{\mu} + \frac{1}{\sqrt{\mu}} + 2\right),$$

or

$$y^2 = ax(\mu^{\frac{1}{4}} + \mu^{-\frac{1}{4}})^2.$$

Secondly, if the points are on different sides of the axis they may be represented by (h, k) and $(\mu h, \ -\sqrt{\mu}k)$.

In this case we shall get $y^2 = -ax(\mu^{\frac{1}{4}} - \mu^{-\frac{1}{4}})^2$.

25. Let the lines make angles α and $90^0 - \alpha$, on opposite sides of the axis: let r_1 and r_2 be the lengths of these lines.

Then, by Art. 155, $\qquad r_1 = \dfrac{4a\cos\alpha}{\sin^2\alpha}$, and $r_2 = \dfrac{4a\sin\alpha}{\cos^2\alpha}$:

but area of triangle $= \frac{1}{2}r_1r_2 = \dfrac{16a^2\sin\alpha\,.\cos\alpha}{2\sin^2\alpha\,.\cos^2\alpha} = \dfrac{16a^2}{\sin 2\alpha}$.

The least value of this is when $\sin 2\alpha = 1$, or when $2\alpha = 90^0$, in which case $\alpha = 45^0$.

In this case the area is $16a^2$.

25. (*Aliter.*) Let the equations to the two lines be $y = mx$ and $y = -\dfrac{1}{m}x$; and let them cut the curve at P and Q.

Combining these with the equation $y^2 = 4ax$, we find that the co-ordinates of P are $\left(\dfrac{4a}{m^2}, \dfrac{4a}{m}\right)$; and the co-ordinates of Q are $(4am^2, \ -4am)$.

Hence $AP = \dfrac{4a}{m^2}\sqrt{1+m^2}$, and $AQ = 4am\sqrt{1+m^2}$.

Hence area of triangle $= \frac{1}{2}AP\,.\,AQ = \dfrac{16a^2(1+m^2)}{2m}$.

Hence we have to find the minimum value of $\dfrac{1+m^2}{2m}$.

Let $$\frac{1+m^2}{2m}=y,$$

$$\therefore\ m^2-2my+y^2=y^2-1;$$

Here the left-hand side is a perfect square, and is therefore positive: hence the right-hand side is positive, so that y^2 is not less than 1.

Hence the least value of y is 1.

In this case the area is $16a^2$.

If $y=1$, $$\frac{m^2+1}{2m}=1,$$

whence we get $m=1=\tan 45^0$.

26. As in the previous example $r=\dfrac{4a\cos\alpha}{\sin^2\alpha}$, and $r'=\dfrac{4a\sin\alpha}{\cos^2\alpha}$.

Hence $$rr'=\frac{16a^2}{\sin\alpha\,.\,\cos\alpha}=\frac{16a^2(\sin^2\alpha+\cos^2\alpha)}{\sin\alpha\,.\,\cos\alpha}=16a^2(\tan\alpha+\cot\alpha).$$

But $$\frac{r'}{r}=\frac{\sin^3\alpha}{\cos^3\alpha}=\tan^3\alpha.$$

$$\therefore\ rr'=16a^2\left(\frac{r'^{\frac{1}{3}}}{r^{\frac{1}{3}}}+\frac{r^{\frac{1}{3}}}{r'^{\frac{1}{3}}}\right).$$

$$\therefore\ r^{\frac{4}{3}}r'^{\frac{4}{3}}=16a^2(r^{\frac{2}{3}}+r'^{\frac{2}{3}}).$$

27. The rectangular equation to the curve with the given origin and axis is $y^2=4a(x-a)$:

But (Art. 8) $x=r\cos\theta$ and $y=r\sin\theta$;

Hence the equation becomes $r^2\sin^2\theta-4ar\cos\theta+4a^2=0$.

Multiply the whole equation by $\sin^2\theta$, and it may be written

$$r^2\sin^4\theta-4ar\sin^2\theta\,.\,\cos\theta+4a^2\cos^2\theta=4a^2(\cos^2\theta-\sin^2\theta)=4a^2\,.\,\cos 2\theta,$$

$$\therefore\ r\sin^2\theta-2a\cos\theta=\pm 2a\sqrt{\cos 2\theta}$$

$$\therefore\ r\sin^2\theta=2a\cos\theta\pm 2a\sqrt{\cos 2\theta}.$$

The equation may also be obtained directly as follows:

In the figure to Art. 125, let $OP=r$, $POM=\theta$;

Now $$SP=PN=OM:$$

$$\therefore\ SP^2=OM^2,$$

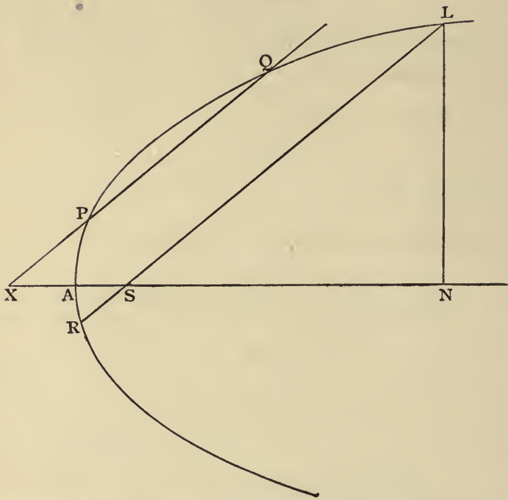
or $$OP^2+OS^2-2OP\,.\,OS\,.\cos\theta=r^2\cos^2\theta,$$

$$\therefore\ r^2+4a^2-4ar\cos\theta=r^2\cos^2\theta,$$

$$\therefore\ r^2\sin^2\theta-4ar\cos\theta+4a^2=0.$$

28. Take X as the pole as in the previous example:

Then $$r^2 - 4ar \cdot \frac{\cos\theta}{\sin^2\theta} + \frac{4a^2}{\sin^2\theta} = 0;$$



Hence (Introd. § II.) $$XP \cdot XQ = \frac{4a^2}{\sin^2\theta}.$$

Again, if we take S as the pole, we have

$$SL = \frac{2a}{1-\cos\theta}, \text{ by Art. 154.}$$

Similarly $$SR = \frac{2a}{1+\cos\theta}:$$

$$\therefore\ SL \cdot SR = \frac{4a^2}{1-\cos^2\theta} = \frac{4a^2}{\sin^2\theta} = XP \cdot XQ.$$

29. In the figure to Art. 125, let $OP = r$, $PON = \theta$.

Now $$SP^2 = PN^2,$$

$$\therefore\ OP^2 + OS^2 - 2OP \cdot OS \cdot \sin\theta = r^2 \sin^2\theta,$$

$$\therefore\ r^2 + 4a^2 - 4ar \cdot \sin\theta = r^2 \sin^2\theta,$$

$$\therefore\ r^2 \cos^2\theta - 4ar \sin\theta + 4a^2 = 0.$$

Multiply the whole equation by $\cos^2\theta$,

$$\therefore\ r^2 \cos^4\theta - 4ar \cdot \sin\theta \cdot \cos^2\theta + 4a^2 \sin^2\theta = 4a^2 (\sin^2\theta - \cos^2\theta),$$

$$\therefore\ r \cos^2\theta - 2a \sin\theta = \pm 2a\sqrt{-\cos 2\theta}.$$

30. Let each chord make an angle of θ with axis; and let any one chord be divided at (x', y') so that the rectangle of its segments is constant.

By Art. 147, the equation determining the lengths of the two segments is

$$r^2 + \frac{2r(y' \sin\theta - 2a\cos\theta)}{\sin^2\theta} + \frac{y'^2 - 4ax'}{\sin^2\theta} = 0.$$

Hence, if the rectangle of the two segments is constant we have

$$\frac{y'^2 - 4ax'}{\sin^2\theta} = \text{constant}, \qquad \text{(Introd. § II.)}$$

Hence $y'^2 - 4ax' =$ a constant; that is to say, the locus of (x', y') is a parabola.

31. Draw CN perpendicular to AB; take A as origin and AB as axis of x; let the co-ordinates of C be (x, y).

Let a, b, c be the sides of the triangle.

Now $$\tan A \,.\, \tan\frac{B}{2} = 2, \quad \therefore \tan A \,.\, \frac{1-\cos B}{\sin B} = 2,$$

$$\therefore \frac{y}{x} \,.\, \frac{1 - \frac{c-x}{a}}{\frac{y}{a}} = 2;$$

$$\therefore x = a - c.$$

$$\therefore (x+c)^2 = a^2 = y^2 + (c-x)^2$$
$$= x^2 + y^2 + c^2 - 2cx.$$

This reduces to $y^2 = 4cx$, which is a parabola whose vertex is A, and focus B.

31. (*Aliter.*) After obtaining the condition $x = a - c$ we might proceed as follows: produce BA to O, so that $AO = AB = c$;

Hence the perpendicular from C on the line through O at right angles to $OB = c + x$, and this equals a by the above condition, that is to say it equals CB. Hence C is on a parabola whose focus is B, and directrix is the line through O at right angles to OB.

32. By Example 5 of this chapter it is seen that the tangents at the ends of the latus rectum meet at the foot of the directrix. Now the equation to the curve with this origin and with the directrix as axis of y is (Art. 125) $y^2 = 4a(x - a)$.

In the present case, our new axes are the old ones turned through 45^0 in negative direction: hence (Art. 81) $x = (x' + y') \,.\, \frac{1}{\sqrt{2}}$,

and $$y = (y' - x') \,.\, \frac{1}{\sqrt{2}}.$$

Substitute these in the original equation and we get

$$(y'-x')^2=4a\sqrt{2}\,(y'+x')-8a^2,$$

$$\therefore\ (y'+x')^2-4a\sqrt{2}(y'+x')+8a^2=4x'y',$$

$$\therefore\ y'+x'-2\sqrt{2}\,.\,a=\pm 2\sqrt{(x'y')},$$

$$\therefore\ y'\pm 2\sqrt{(x'y')}+x'=2\sqrt{2}a,$$

$$\therefore\ \sqrt{y'}\pm\sqrt{x'}=\pm\sqrt{(2a\sqrt{2})}.$$

Since with these axes all co-ordinates are positive, it is evident that all the roots in the final equation indicate *possible* operations.

33. The axes in this example are parallel to those in the previous, the axis of y being the same in both examples. Let x'', y'' be the co-ordinates of a point referred to our present axes, then using the notation of the previous example we have $x''=x'$ and $y''+2a\sqrt{2}=y'$.

Substituting in the equation of last example, we get

$$(y''-2a\sqrt{2}-x'')^2=4a\sqrt{2}\,(y''+2a\sqrt{2}+x'')-8a^2,$$

which reduces to $(y''-x'')^2-8ax''\sqrt{2}=0$.

34. The co-ordinates of the centre are $\frac{a+x'}{2}$, $\frac{y'}{2}$, and radius

$$=\tfrac{1}{2}SP=\frac{a+x'}{2}.$$

Hence the equation is

$$\left(x-\frac{a+x'}{2}\right)^2+\left(y-\frac{y'}{2}\right)^2=\left(\frac{a+x'}{2}\right)^2,$$

or

$$x^2-x\,(a+x')+y^2-yy'+ax'=0.$$

35. Solving the equation in previous example with the equation $x=0$, we get the quadratic $y^2-yy'+\frac{y'^2}{4}=0$, which gives two *coincident* values of y, namely $\frac{y'}{2}$; hence the tangent at vertex touches the circle at the point $\left(0,\ \frac{y'}{2}\right)$.

36. Solving the equations $y=m(x-a)$ and $y^2=4ax$ simultaneously, we get the quadratic $x^2-x\left(2a+\frac{4a}{m^2}\right)+a^2=0$.

Hence (Introd. § II.) $$x'+x''=2a+\frac{4a}{m^2},$$

$$x'x''=a^2.$$

Similarly from the quadratic involving y we get the other two results.

[Note. Since $y'y''=-4a^2$; $\therefore\ -\frac{y'}{2a}=\frac{2a}{y''}$; hence the normals at (x',y') and (x'',y'') are at right angles. So also are the tangents at these points. Hence the normals intersect on the circle whose diameter is the chord, and so do the tangents. See Art. 156.]

37. Let $y=m(x-a)$ be the focal chord, and (x', y'), (x'', y'') its extremities. Then, by the previous example, $x'+x''=2a+\frac{4a}{m^2}$, but the abscissa of the middle point of the chord is

$$\tfrac{1}{2}(x'+x'')=a+\frac{2a}{m^2}.$$

Similarly the ordinate of middle point $=\frac{2a}{m}$.

Also, square of diameter of circle

$$=(x'-x'')^2+(y'-y'')^2=(x'+x'')^2-4x'x''+(y'+y'')^2-4y'y''$$

(Introd. § v.).

$$=4a^2+\frac{16a^2}{m^2}+\frac{16a^2}{m^4}-4a^2+\frac{16a^2}{m^2}+16a^2=16a^2\left(1+\frac{2}{m^2}+\frac{1}{m^4}\right).$$

Hence square of radius

$$=4a^2\left(1+\frac{2}{m^2}+\frac{1}{m^4}\right);$$

therefore equation to required circle is

$$\left(x-a-\frac{2a}{m^2}\right)^2+\left(y-\frac{2a}{m}\right)^2=4a^2\left(1+\frac{2}{m^2}+\frac{1}{m^4}\right);$$

or

$$x^2+y^2-2ax\left(1+\frac{2}{m^2}\right)-\frac{4ay}{m}-3a^2=0.$$

[*Note.* The length of the diameter might be got by Art. 129; it is evident that the two portions of the chord are $a+x'$ and $a+x''$, hence diameter

$$=2a+x'+x''=2a+2a+\frac{4a}{m^2}=4a\left(1+\frac{1}{m^2}\right).]$$

37. (*Aliter.*) By using the result of Ex. 5, Chap. VI., we at once get the required equation to the circle in the shape

$$x^2+y^2-x(x'+x'')-y(y'+y'')+x'x''+y'y''=0.$$

Substitute the values given in Ex. 36, and we at once obtain the result.

38. In the equation of the previous example, put $x=-a$, and we get

$$y^2-\frac{4ay}{m}+\frac{4a^2}{m^2}=0.$$

This gives two identical values of y, hence the circle *touches* the directrix.

39. The equation to the tangent at (x', y') with vertex as origin is

$$yy'=2a(x+x').$$

Transfer the origin to the point $(a, 0)$, and the equation becomes

$$yy'=2a(x+x'+2a).$$

40. The equation to the tangent with vertex as origin is

$$y = mx + \frac{a}{m}.$$

Transfer the origin to the point $(a, 0)$, and the equation becomes

$$y = m(x+a) + \frac{a}{m}.$$

41. Let the latus rectum of one parabola be $4a$, and of the other $4a'$. Take the focus as origin, and the axis as axis of x.

The equation to one tangent being

$$y = m(x+a) + \frac{a}{m},$$

the equation to the other will be

$$y = -\frac{1}{m}(x+a') - a'm.$$

Subtract one equation from the other, and we get

$$0 = \left(m + \frac{1}{m}\right)x + \left(m + \frac{1}{m}\right)a + \left(m + \frac{1}{m}\right)a'.$$

Divide by $m + \frac{1}{m}$, and the equation becomes

$$0 = x + a + a';$$

hence the locus is a straight line parallel to the axis of y.

42. It is evident that the equation to a tangent to $y^2 = 8a(x-c)$ will be

$$y = m(x-c) + \frac{2a}{m}.$$

To find the co-ordinates of the points (x_1, y_1) and (x_2, y_2) where this cuts the curve $y^2 = 4ax$, we must solve the equations simultaneously. We shall get the quadratic

$$x^2 - 2cx + \left(c - \frac{2a}{m^2}\right)^2 = 0.$$

Hence $$x_1 + x_2 = 2c, \text{ and } \frac{x_1 + x_2}{2} = c.$$

That is, the abscissa of middle point of the chord is c.

43. The equation to a normal is $y = mx - 2am - am^3$; if this passes through (h, k), we get

$$k = mh - 2am - am^3.$$

This is a cubic equation to find m, and we shall therefore get three values, giving three normals.

44. In the equation of the previous example the term containing the second power of m is wanting; hence the sum of the roots is zero. Hence, bearing in mind that $m=-\frac{y'}{2a}$, we get

$$-\frac{y_1}{2a}-\frac{y_2}{2a}-\frac{y_3}{2a}=0,$$

or

$$y_1+y_2+y_3=0.$$

Secondly, suppose the circle $x^2+y^2+Ax+By+C=0$ to cut the parabola in the four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4); solving this equation simultaneously with $y^2=4ax$, we get

$$\frac{y^4}{16a^2}+y^2+\frac{Ay^2}{4a}+By+C=0.$$

In this equation the term containing y^3 is wanting; hence

$$y_1+y_2+y_3+y_4=0.$$

But $y_1+y_2+y_3=0$; therefore $y_4=0$. Similarly $x_4=0$. Hence the circle goes through the origin.

45. Let the equation to one normal be

$$y=mx-2am-am^3,$$

then the other normal at right angles to it is

$$y=-\frac{x}{m}+\frac{2a}{m}+\frac{a}{m^3}.$$

If we eliminate m between these two equations, we get the locus of the required point.

To perform the elimination, proceed as follows:

Multiply the second equation by m^2, and add the result to the first equation, we get

$$y(1+m^2)=\frac{a}{m}(1-m^4);$$

$$\therefore \quad y=\frac{a}{m}(1-m^2)=a\left(\frac{1}{m}-m\right).$$

Again, subtract the one equation from the other, and divide by $m+\frac{1}{m}$: we get

$$x-2a=a\left(m^2-1+\frac{1}{m^2}\right);$$

$$\therefore \quad ax-2a^2=a^2\left\{m^2-2+\frac{1}{m^2}+1\right\}$$

$$=y^2+a^2;$$

$$\therefore \quad y^2=ax-3a^2$$

is the locus, which is a parabola.

45. (*Aliter.*) In the method of solution just given, we are dependent upon our skill in eliminating m; in case we failed in this point, the question could be treated thus.

The equation to a normal is

$$y = mx - 2am - am^3.$$

If (h, k) be the point whose locus is required, we have

$$k = mh - 2am - am^3,$$

or

$$m^3 + \left(2 - \frac{h}{a}\right)m + \frac{k}{a} = 0.$$

Let the roots of this be m_1, m_2, m_3: hence, by the Theory of Equations,

$$m_1 + m_2 + m_3 = 0; \quad m_1m_2 + m_1m_3 + m_2m_3 = 2 - \frac{h}{a}; \quad m_1m_2m_3 = -\frac{k}{a}.$$

But since two of the normals are at right angles, we have

$$m_1m_2 = -1 \quad \text{(Art. 42)}.$$

Hence the equations become

$$m_1 + m_2 = -m_3; \quad m_3(m_1 + m_2) = 3 - \frac{h}{a}; \quad m_3 = \frac{k}{a};$$

$$\therefore \quad m_3(-m_3) = 3 - \frac{h}{a};$$

$$\therefore \quad \frac{h}{a} - 3 = m_3^2 = \frac{k^2}{a^2};$$

$$\therefore \quad k^2 = ah - 3a^2.$$

Hence the point is on the locus

$$y^2 = ax - 3a^2.$$

45. (*Aliter.*) A third method of solution is as follows:

If R be a point such that RP, RQ are two normals at right angles, it is required to find the locus of R.

Now, by the note to Ex. 36 it is evident that PQ is a focal chord.

And the diagonal ZR bisects PQ, and is therefore a diameter, that is to say it is parallel to the axis (Art. 153).

Let the angle $PSM = \theta$.

Now, by Art. 154

$$SP = \frac{2a}{1 - \cos\theta}; \quad SQ = \frac{2a}{1 + \cos\theta};$$

$$\therefore \quad SL = SP - \tfrac{1}{2}QP = \frac{2a\cos\theta}{\sin^2\theta};$$

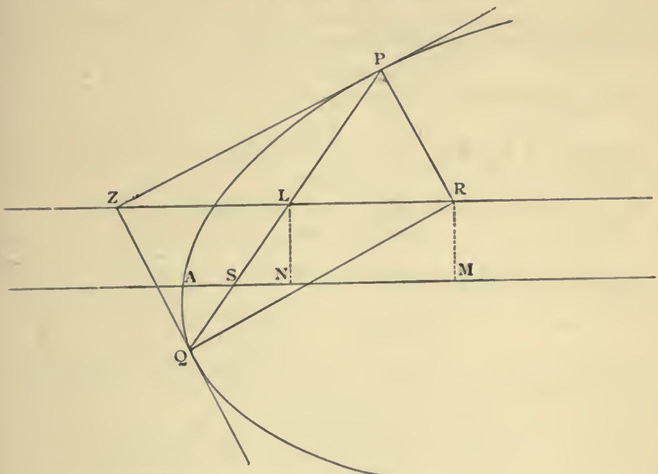
$$\therefore \quad LN = 2a\cot\theta.$$

But $\qquad RM = LN, \quad \therefore \; RM = 2a\cot\theta.$

Also $ZR=PQ$ since each of these is a diameter of the circle $PZQR$;

$$\therefore \quad ZR=SP+SQ=\frac{4a}{\sin^2\theta}=4a(1+\cot^2\theta);$$

$$\therefore \quad AM=3a+4a\cot^2\theta.$$



Hence if (x, y) be the co-ordinates of R, we have

$$y=2a\cot\theta, \quad x=3a+4a\cot^2\theta;$$

$$\therefore \quad y^2=4a^2\cot^2\theta=a(x-3a).$$

46. Taking the focus as origin, the two parabolas will be

$$y^2=4a(x+a) \text{ and } x^2=4a(y+a).$$

Subtracting, $\qquad y^2-x^2=4a(x-y);$

therefore either $\qquad x=y \text{ or } x+y=-4a.$

One of these two lines must be the common chord, but as the common chord will obviously pass through the origin (as will be seen at once from considerations of symmetry if a figure be drawn), it must be the former one, namely $x=y$.

Let this meet the curves at the points (x_1, y_1) and (x_2, y_2); solving the equations $x=y$ simultaneously with the equation to one of the curves, we get the quadratic

$$x^2=4ax+4a^2;$$

$$\therefore \quad x_1+x_2=4a, \text{ and } x_1x_2=-4a^2.$$

Similarly $y_1+y_2=4a$, and $y_1y_2=-4a^2$;

therefore common chord

$$=\sqrt{\{(x_1+x_2)^2-4x_1x_2+(y_1+y_2)^2-4y_1y_2\}}$$
$$=8a.$$

Also if in the equation of Ex. 40 we put $m=1$, we get the tangent to the first parabola parallel to the common chord to be

$$y=(x+a)+a \quad \text{or} \quad y=x+2a.$$

Where this touches the curve $y^2=4a(x+a)$, we have

$$x=0, \quad y=2a.$$

Similarly the tangent to the other curve parallel to the common chord touches at the point $x=2a$, $y=0$.

And the length of the line joining these two points is

$$\sqrt{(4a^2+4a^2)}=2a\sqrt{2}.$$

47. The equation to the chord of contact is $ky-2ax-2ah=0$; the perpendicular from (h, k) on this is

$$\frac{k^2-2ah-2ah}{\sqrt{(k^2+4a^2)}}=\frac{k^2-4ah}{\sqrt{(k^2+4a^2)}}.$$

48. Combining the equation to the chord of contact with the equation to the curve, we get

$$y^2-2ky+4ah=0.$$

Hence if the points of contact are (x_1, y_1) and (x_2, y_2), we have

$$y_1+y_2=2k, \quad y_1y_2=4ah.$$

Similarly $x_1+x_2=\dfrac{k^2}{a}-2h$; $x_1x_2=h^2$.

Hence, square of common chord

$$=(x_1-x_2)^2+(y_1-y_2)^2$$
$$=(x_1+x_2)^2-4x_1x_2+(y_1+y_2)^2-4y_1y_2$$
$$=\frac{(k^2-4ah)(k^2+4a^2)}{a^2};$$

therefore common chord

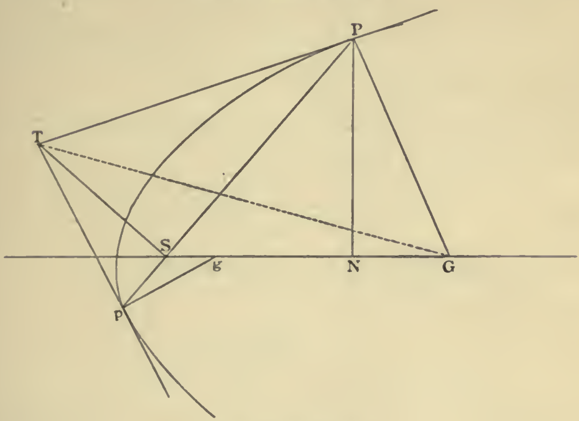
$$=\frac{(k^2-4ah)^{\frac{1}{2}}(k^2+4a^2)^{\frac{1}{2}}}{a}.$$

49. The area of triangle is half the product of the chord of contact and the perpendicular on it from (h, k)

$$=\tfrac{1}{2}\cdot\frac{k^2-4ah}{\sqrt{(k^2+4a^2)}}\cdot\frac{\sqrt{(k^2-4ah)}\cdot\sqrt{(k^2+4a^2)}}{a} \text{ by Ex. 47, 48}$$
$$=\frac{(k^2-4ah)^{\frac{3}{2}}}{2a}.$$

50. Let the co-ordinates of P and Q be

$$(x', y') \text{ and } (x'', y'').$$



Now
$$PG^2 = PN^2 + NG^2$$
$$= y'^2 + 4a^2$$
$$= 4a(x' + a).$$

Similarly
$$pg^2 = 4a(x'' + a);$$

$$\therefore \quad \frac{1}{PG^2} + \frac{1}{pg^2} = \frac{1}{4a}\left\{\frac{1}{x' + a} + \frac{1}{x'' + a}\right\}$$

$$= \frac{1}{4a}\left\{\frac{x' + x'' + 2a}{x'x'' + a(x' + x'') + a^2}\right\}.$$

Using the results of Ex. 36, this becomes

$$\frac{1}{4a}\left\{\frac{4a + \frac{4a}{m^2}}{a^2 + 2a^2 + \frac{4a^2}{m^2} + a^2}\right\} = \frac{1}{4a^2}, \text{ a constant.}$$

Again, by similar triangles $\dfrac{TS}{TP} = \dfrac{GN}{GP}$;

$$\therefore \quad \frac{PG}{PT} = \frac{GN}{ST} = \frac{2a}{ST};$$

$$\therefore \quad \tan PTG = \frac{2a}{ST}.$$

Similarly $$\tan pTg = \frac{2a}{ST};$$

$$\therefore\ PTG = pTg.$$

51. Let the equation to the outer parabola be $y^2 = 4ax$, and to the inner one be $$y^2 = 4a(x - c).$$

Let the co-ordinates of O be (x', y').

Let θ and $90^0 + \theta$ be the inclinations of POp and QOq to the axis.

Then (by Art. 147) PO, Op are the roots of

$$r^2 \sin^2\theta + 2r(y' \sin\theta - 2a\cos\theta) + y'^2 - 4ax' = 0.$$

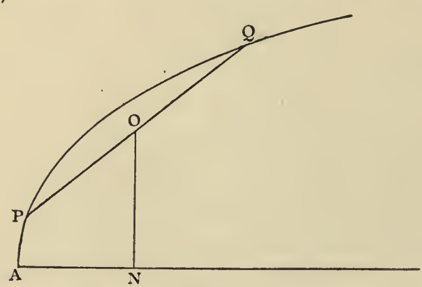
Hence $$PO \,.\, Op = \frac{y'^2 - 4ax'}{\sin^2\theta},$$

which is numerically $= \dfrac{4ac}{\sin^2\theta}$, since O is on inner curve.

Similarly $QO \,.\, Oq$ is numerically $= \dfrac{4ac}{\cos^2\theta}$;

$$\therefore\ \frac{1}{PO \,.\, Op} + \frac{1}{QO \,.\, Oq} = \frac{\cos^2\theta + \sin^2\theta}{4ac} = \frac{1}{4ac}.$$

52. Let PQ be the chord, and O its middle point. Let the co-ordinates of O be (x', y').



Then the equation of Art. 147, namely,

$$r^2 \sin^2\theta + 2r(y' \sin\theta - 2a\cos\theta) + y'^2 - 4ax' = 0,$$

is to have its roots each numerically equal to c but of opposite signs; hence the sum of the roots is zero;

$$\therefore\ y' \sin\theta = 2a\cos\theta.$$

But $y' = c$, since the circle touches the axis;

$$\therefore\ c\sin\theta = 2a\cos\theta,$$

or $$\tan\theta = \frac{2a}{c}.$$

53. Let (x', y') and (x'', y'') be the points of contact. Then, as in Ex. 48, we have $y'+y''=2k$; $y'y''=4ah$.

But
$$\tan\theta=\frac{2a}{y'}, \qquad \tan\theta'=\frac{2a}{y''};$$

$$\therefore \tan\theta+\tan\theta'=\frac{2a(y'+y'')}{y'y''}=\frac{4ak}{4ah}=\frac{k}{h}.$$

And
$$\tan\theta\,.\,\tan\theta'=\frac{4a^2}{y'y'},=\frac{4a^2}{4ah}=\frac{a}{h}.$$

54. With the notation of the previous example, we have $\theta+\theta'=$ constant $=\alpha$;

$$\therefore \tan\alpha=\tan(\theta+\theta')=\frac{\tan\theta+\tan\theta'}{1-\tan\theta\,.\,\tan\theta'}$$

$$=\frac{\dfrac{k}{h}}{1-\dfrac{a}{h}}=\frac{k}{h-a}.$$

$$\therefore k=(h-a)\tan\alpha.$$

Consequently (h, k) lies on the straight line, $y=(x-a)\tan\alpha$ through the focus.

55. Let the two tangents be

$$y-k=\tan\theta(x-h) \text{ and } y-k=\tan\theta'(x-h).$$

These can be combined in the one equation

$$(y-k)^2-(\tan\theta+\tan\theta')(x-h)(y-k)+\tan\theta\,.\,\tan\theta'(x-h)^2=0.$$

Using the results of Ex. 53, this becomes

$$h(y-k)^2-k(x-h)(y-k)+a(x-h)^2=0.$$

Multiply this equation by $4a$ and it produces the second shape.

[*Note.* The second shape can be also readily obtained by the method used in Chap. VI. Ex. 35. *Aliter.*]

56. The equation to the chord of contact is $ky-2ax=2ah$; and the equation to the curve is $y^2=4ax$; hence the equation $2ah(y^2)=4ax(ky-2ax)$ or $hy^2=2x(ky-2ax)$ is satisfied at the intersection of the curve and the chord of contact. In other words this equation represents some locus through the points of contact.

Also writing the equation in the form $hy^2-2kxy+4ax^2=0$, and bearing in mind that $k^2>4ah$ (Art. 127), we see by Art. 61 that it represents two straight lines through the origin.

57. Combining the two equations we get $x=\frac{a}{m_1m_2}$ and $y=\frac{a}{m_1}+\frac{a}{m_2}$.

Let the third tangent be $y=m_3x+\frac{a}{m_3}$; then the equation to the required perpendicular is

$$y-\frac{a}{m_1}-\frac{a}{m_2}=-\frac{1}{m_3}\left(x-\frac{a}{m_1m_2}\right).$$

To find where this cuts the directrix, put $x=-a$;

$$\therefore\ y=\frac{a}{m_1}+\frac{a}{m_2}+\frac{a}{m_3}+\frac{a}{m_1m_2m_3}.$$

58. Let the triangle be formed by the three tangents

$$y=m_1x+\frac{a}{m_1};\quad y=m_2x+\frac{a}{m_2};\quad y=m_3x+\frac{a}{m_3}.$$

The perpendicular from the intersection of any two of these on the third cuts the directrix at the point whose ordinate is, by previous example,

$$\frac{a}{m_1}+\frac{a}{m_2}+\frac{a}{m_3}+\frac{a}{m_1m_2m_3};$$

that is to say, all the perpendiculars intersect at the same point.

CHAPTER IX.

1. Here $$\frac{b^2}{a^2}=\frac{2}{3},\ \therefore\ e^2=1-\frac{2}{3}=\frac{1}{3},\ \therefore\ e=\frac{1}{\sqrt{3}}.$$

2. In the figure to Art. 162, the co-ordinates of L are $(ae, a-ae^2)$; hence the required equation is

$$a^2y(a-ae^2)+b^2x(ae)=a^2b^2,$$

or $$a^3(1-e^2)y+a^3e(1-e^2)x=a^4(1-e^2),$$

or $$y+ex=a.$$

The intercepts on the axes are $\frac{a}{e}$ and a.

3. The normal being perpendicular to the tangent, whose equation was found in previous example, its equation is

$$y-(a-ae^2)=\frac{1}{e}(x-ae),$$

or $$y+ae^2=\frac{x}{e}.$$

4. In the equation to the normal make $x=0$, $\therefore\ y=-ae^2$; but by the question the intercept is to be $-b$;

$$\therefore\ ae^2=b,\quad \therefore\ e^4=\frac{b^2}{a^2}=1-e^2,\quad \therefore\ e^4+e^2=1.$$

This determines the excentricity.

5. The equation to $A'B$ is $y=\frac{b}{a}(x+a)$, and the equation to CL is $y=\frac{b^2}{a^2e}x$.

If these are parallel $\frac{b}{a}=\frac{b^2}{a^2e}$ or $e=\frac{b}{a}$:

$$\therefore\ e^2=\frac{b^2}{a^2}=1-e^2;\quad \therefore\ 2e^2=1;\quad \therefore\ e=\frac{1}{\sqrt{2}}.$$

6. The equation to $B'H$ is $y=\frac{b}{ae}(x-ae)$.

Solving this simultaneously with the equation to the ellipse we get $x=0$ or $\frac{2ae}{1+e^2}$; hence the required abscissa is $\frac{2ae}{1+e^2}$.

7. The equation to AL is $y=-(1+e)(x-a)$.

Tangent of angle included between AL and tangent at L is

$$\frac{1+e-e}{1+e(1+e)}=\frac{1}{1+e+e^2}.$$

8. The equation to PH is $y=\frac{y'}{x'-ae}(x-ae)$; solving this simultaneously with the equation to the ellipse we get

$$a^2y'^2(x-ae)^2+b^2x^2(x'-ae)^2=a^2b^2(x'-ae)^2,$$

or $$(a^2b^2-b^2x'^2)(x-ae)^2+b^2x^2(x'-ae)^2=a^2b^2(x'-ae)^2.$$

In this equation we shall find that the product of the roots is

$$\frac{a(2aex'-x'^2-e^2x'^2)}{a+ae^2-2ex'};$$

but one root is x', hence the other root is $\frac{2a^2e-ax'(1+e^2)}{a(1+e^2)-2ex'}$.

9. The equation to tangent at (x', y') is $y=-\frac{b^2x'}{a^2y'}x+\frac{b^2}{y'}$; if this is equally inclined to the axes $\frac{b^2x'}{a^2y'}=\pm 1$. From this equation and the equation to the ellipse we get $x'=\pm\frac{a^2}{\sqrt{(a^2+b^2)}}$ and $y'=\pm\frac{b^2}{\sqrt{(a^2+b^2)}}$.

10. The intercepts made by the tangent are $\frac{a^2}{x'}$, and $\frac{b^2}{y'}$; if $\frac{a^2}{x'} : \frac{b^2}{y'} :: a : b$, we have $ay' = bx'$. Hence by the equation to the curve we get

$$x' = \pm \frac{a}{\sqrt{2}}, \qquad y' = \pm \frac{b}{\sqrt{2}}.$$

11. Using the figure to Art. 162,

$$\tan APA' = \tan (APM + A'PM) = \frac{\frac{AM}{PM} + \frac{A'M}{PM}}{1 - \frac{AM \cdot A'M}{PM^2}}$$

$$= \frac{PM \cdot AA'}{PM^2 - (CA^2 - CM^2)} = \frac{2ay}{y^2 - a^2 + x^2} = \frac{2ab^2y}{b^2y^2 - a^2y^2} = -\frac{2b^2}{ae^2y}.$$

12. In Art. 178 it is proved that $\tan GPH = \frac{eay'}{b^2}$, and HPZ is the complement of GPH, $\therefore \tan HPZ = \frac{b^2}{eay'}$.

13. $$PC^2 = y'^2 + x'^2 = y'^2 + a^2 - \frac{a^2}{b^2} y'^2 = a^2 - \frac{a^2e^2}{b^2} y'^2 = a^2 - b^2 \cot^2 \phi.$$

14. Let AQ be perpendicular to AP, and $A'Q'$ to $A'P$.

The equation to AP is $y = \frac{y'}{x' - a}(x - a)$:

$\therefore$ equation to AQ is $y = -\frac{x' - a}{y'}(x - a)$.

Similarly the equation to $A'Q'$ is $y = -\frac{x' + a}{y'}(x + a)$.

If between the equations we eliminate x', y', we shall get the locus required.

Multiply the equations together, then

$$y^2 = \frac{x'^2 - a^2}{y'^2}(x^2 - a^2),$$

or

$$y^2 = -\frac{a^2}{b^2}(x^2 - a^2),$$

or

$$\frac{b^2y^2}{a^4} + \frac{x^2}{a^2} = 1.$$

Hence the locus is an ellipse, whose axes are $\frac{2a^2}{b}$ and $2a$.

15. Let the co-ordinates of P be (x', y') and of Q be (x', y''); since Q is on the tangent at L we have $y'' + ex' = a$;

$$\therefore\ y'' = a - ex' = HP, \text{ by Art. 166.}$$

16. The four tangents are $y+ex=a$; $y-ex=a$; $-y+ex=a$; $-y-ex=a$. Whatever be the value of e it is evident that the first two pass through the fixed point $(0, a)$, and the other two through the fixed point $(0, -a)$.

17. Let S be the common focus, P and Q the points of intersection. Take C as origin.

The equation to one parabola is $y^2=4(a-ae)(x+a)$ and to the other

$$y^2=4(a+ae)(-x+a).$$

Solving these simultaneously we get $y=\pm 2b$, and $x=ae$.

Hence P and Q are vertically above and below the other focus, and at a distance apart $=4b$.

If θ, θ' be the inclinations of the two tangents at P, we have

$$\tan\theta=\frac{\text{semi-latus rectum}}{y'}=\frac{2(a-ae)}{2b}=\frac{a-ae}{b}.$$

Similarly $\quad \tan\theta'=-\dfrac{a+ae}{b}.$

$$\therefore\ 1+\tan\theta\,.\tan\theta'=1-\frac{a^2-a^2e^2}{b^2}=1-1=0.$$

Hence, by Art. 42, the tangents are at right angles.

18. See figure to Art. 175,

$$PG'^2=PN^2+NG'^2=x^2+(y+c)^2.$$

But, by Art. 176, $\qquad c=\dfrac{a^2e^2}{b^2}y,$

$$\therefore\ y+c=\frac{a^2}{b^2}y.$$

$$\therefore\ PG'^2=x^2+\frac{a^4}{b^4}y^2=a^2-\frac{a^2}{b^2}y^2+\frac{a^4}{b^4}y^2=a^2+\frac{a^4e^2y^2}{b^4}=a^2+\frac{c^2}{e^2}.$$

19. Let the angle $CHM=\alpha$.

Hence the co-ordinates of M are $(0, ae\tan\alpha)$.

Also $NA=a\sec\alpha$.

Hence the equation to the circle is

$$x^2+(y-ae\tan\alpha)^2-a^2\sec^2\alpha=0.$$

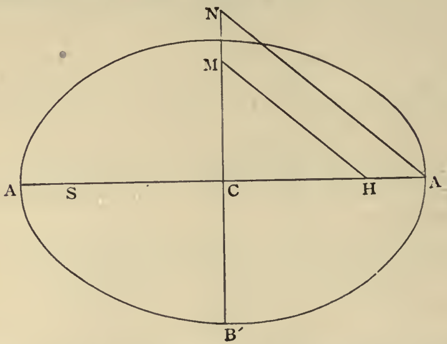
Combining this with the equation to the ellipse we get

$$a^2e^2y^2+2aeb^2y\,.\tan\alpha+b^4\tan^2\alpha=0.$$

This equation has *equal* roots; hence if these roots be possible they give the value of the ordinate of the point where the ellipse and circle *touch*.

This ordinate is $-\dfrac{b^2\tan\alpha}{ae}$.

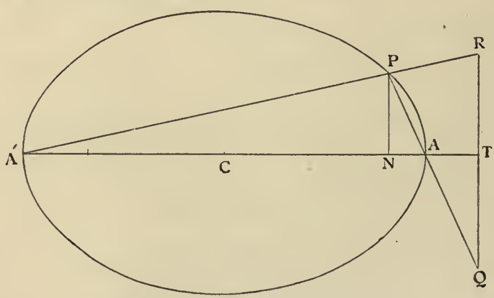
But in the ellipse the greatest ordinate is $\pm b$.



Hence if $\tan\alpha > \frac{ae}{b}$, the above value of the ordinate is impossible; that is to say, the ellipse lies within the circle.

If $\tan\alpha$ is not $> \frac{ae}{b}$, the above value is possible, and the ellipse touches the circle internally.

20. Let PN be the ordinate of P.



Now $\frac{RT}{PN} = \frac{A'T}{A'N} = \frac{a + \frac{a^2}{x}}{a+x} = \frac{a}{x}$.

Similarly $\frac{QT}{PN} = \frac{AT}{AN} = \frac{\frac{a^2}{x} - a}{a-x} = \frac{a}{x}$.

$$\therefore RT = QT.$$

21. The equation to the chord joining $(a\cos\phi,\ b\sin\phi)$ with $(a\cos\phi',\ b\sin\phi')$ is

$$y-b\sin\phi=\frac{b}{a}\cdot\frac{\sin\phi'-\sin\phi}{\cos\phi'-\cos\phi}(x-a\cos\phi)$$

$$=-\frac{b}{a}\cdot\frac{\cos\frac{1}{2}(\phi+\phi')}{\sin\frac{1}{2}(\phi+\phi')}(x-a\cos\phi);$$

$$\therefore\ \frac{y}{b}\cdot\sin\tfrac{1}{2}(\phi+\phi')+\frac{x}{a}\cdot\cos\tfrac{1}{2}(\phi+\phi')=\cos\phi\cdot\cos\tfrac{1}{2}(\phi+\phi')+\sin\phi\cdot\sin\tfrac{1}{2}(\phi+\phi')$$

$$=\cos\tfrac{1}{2}(\phi-\phi').$$

22. Putting $\phi'=\phi$ in the last example we get

$$\frac{y}{b}\sin\phi+\frac{x}{a}\cos\phi=1,$$

as the required equation to the tangent.

23. By the last example the equation to the tangent is

$$y=-\frac{b}{a}\cot\phi\cdot x+\frac{b}{\sin\phi}.$$

The equation to a straight line through $(a\cos\phi,\ b\sin\phi)$ perpendicular to this is

$$y-b\sin\phi=\frac{a\tan\phi}{b}(x-a\cos\phi),$$

or

$$ax\sec\phi-by\operatorname{cosec}\phi=a^2-b^2.$$

24. (See figure to Art. 175.)

The co-ordinates of the middle point of PG being (x, y) and those of P being (x', y') we have $x=\frac{1}{2}(x'+e^2x')$ by Art. 176.

And $$y=\tfrac{1}{2}y'.$$

If we eliminate x', y' from these equations by the help of the condition $a^2y'^2+b^2x'^2=a^2b^2$, we get $\dfrac{4x^2}{a^2(1+e^2)^2}+\dfrac{4y^2}{b^2}=1$;

hence the locus is an ellipse whose axes are $a(1+e^2)$ and b.

Let e' be the excentricity of this ellipse;

$$\therefore\ a^2(1+e^2)^2(1-e'^2)=b^2=a^2(1-e^2);$$

$$\therefore\ (1+e^2)^2(1-e'^2)=1-e^2.$$

25. The tangent at L is $\quad y+ex=a,$

and equation to HB is $\quad aey+bx=aeb.$

Hence, for the co-ordinates of the point of intersection

$$x=\frac{ae(a-b)}{ae^2-b};\ y=\frac{ab(e^2-1)}{ae^2-b}.$$

If the lines are parallel $e=\frac{b}{ae}$, or $e^4=\frac{b^2}{a^2}=1-e^2$.

From this equation e can be found.

[The ellipse is similar to that in example 4.]

26. The tangent at P is $a^2yy'+b^2xx'=a^2b^2$.

Put $x=\frac{a}{e}$, $\therefore\ ET=\frac{ab^2(ae-x')}{a^2ey'}=\frac{ab^2}{a^2e}\cdot\frac{MH}{PM}=\frac{b^2}{ae}\cdot\cot PHS$.

$$\therefore\ ET\propto\cot PHS.$$

Again put $x=-\frac{a}{e}$, $\therefore\ E'T'=\frac{ab^2(ae+x')}{a^2ey'}=\frac{ab^2}{a^2e}\cdot\frac{SM}{PM}=\frac{b^2}{ae}\cdot\cot PSH$.

$$\therefore\ E'T'\propto\cot PSH.$$

27. Let PQ be the chord, the co-ordinates of P and Q being (x_1, y_1) and (x_2, y_2).

$$\therefore\ PQ^2=(x_1-x_2)^2+(y_1-y_2)^2$$
$$=(x_1-x_2)^2+(mx_1-mx_2)^2,$$

since P and Q are on the line $y=mx+c$.

$$\therefore\ PQ^2=(m^2+1)(x_1-x_2)^2,$$
$$=(m^2+1)\{(x_1+x_2)^2-4x_1x_2\}.$$

But, combining the equations to the line and curve together, we get

$$a^2m^2x^2+2a^2mcx+a^2c^2+b^2x^2-a^2b^2=0.$$

$$\therefore\ x_1+x_2=-\frac{2a^2mc}{a^2m^2+b^2};\ x_1x_2=\frac{a^2c^2-a^2b^2}{a^2m^2+b^2}.$$

Substituting these in the value of PQ^2 we get the required result.

If the line is a tangent, $PQ=0$; hence we get

$$c^2=a^2m^2+b^2.$$

28. (Figure to Art. 166.)

The co-ordinates of the centre are $\frac{1}{2}(ae+x')$ and $\frac{1}{2}y'$; and the radius

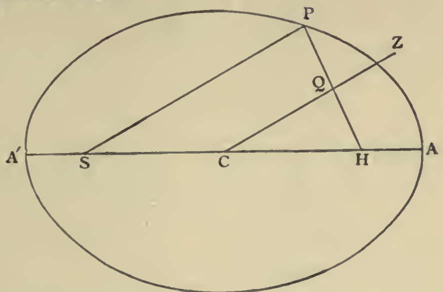
$$=\tfrac{1}{2}HP=\tfrac{1}{2}(a-ex').$$

Hence required equation is

$$x^2+y^2-(ae+x')x-yy'+\left(\frac{ae+x'}{2}\right)^2+\left(\frac{y'}{2}\right)^2-\left(\frac{a-ex'}{2}\right)^2=0,$$

which reduces to $x^2+y^2-(ae+x')x-yy'+aex'=0$.

29. Bisect HP at Q, and produce CQ to Z, so that $QZ=\frac{1}{2}HP$.



Now $CQ=\frac{1}{2}SP$ (Eucl. VI. 4);

$\therefore\ CZ=\frac{1}{2}(SP+HP)=AC$.

Hence Z is on the auxiliary circle; but it is also on the circle whose centre is Q and radius QP; hence the two circles touch at Z.

[*Note.* It is easily seen that Z is the point Z of Art. 180.]

30. The equation to the chord of contact is

$$a^2yk+b^2xh-a^2b^2=0.$$

Hence the perpendicular from S is $\pm\dfrac{b^2aeh+a^2b^2}{\sqrt{(a^4k^2+b^4h^2)}}$;

and the perpendicular from H is $\pm\dfrac{a^2b^2-b^2aeh}{\sqrt{(a^4k^2+b^4h^2)}}$.

If $h<\dfrac{a}{e}$ numerically, that is to say if (h, k) is between the directrices, the numerators of the two above expressions are of the same sign; in this case, the sum of perpendiculars $=\dfrac{2a^2b^2}{\sqrt{(a^4k^2+b^4h^2)}}$.

In any other case, the sum $=\pm\dfrac{2ab^2eh}{\sqrt{(a^4k^2+b^4h^2)}}$.

31. By Ex. 23, the equation to normal at P is

$$ax\sec\phi-by\operatorname{cosec}\phi=a^2-b^2,$$

and the equation to CQ is $\qquad y=x\tan\phi.$

Solving simultaneously, we find the co-ordinates of R to be

$$(a+b)\cos\phi \text{ and } (a+b)\sin\phi.$$

These evidently satisfy the equation $y^2+x^2=(a+b)^2$, which is a circle with centre C and radius $a+b$.

32. Let the semi-axes of one ellipse be a, b, and of the other c, d.

Then the form of a tangent to the first is $y=mx+\sqrt{a^2m^2+b^2}$; and to the second $y=mx+\sqrt{c^2m^2+d^2}$.

If these are identical $a^2m^2+b^2=c^2m^2+d^2$;

$$\therefore\ m^2=\frac{d^2-b^2}{a^2-c^2}.$$

Consequently the common tangents are expressed by

$$y=\pm x\sqrt{\left(\frac{d^2-b^2}{a^2-c^2}\right)}\pm\sqrt{\left(\frac{a^2d^2-b^2c^2}{a^2-c^2}\right)}.$$

Also, if $a^2+b^2=c^2+d^2$, then $m^2=1$.

Hence the common tangents are

$$y=\pm x\pm\sqrt{a^2+b^2}.$$

33. The equation to a tangent is $y=mx+\sqrt{(a^2m^2+b^2)}$, and if it passes through (h, k) we have $k=mh+\sqrt{(a^2m^2+b^2)}$,

or $$m^2(a^2-h^2)+2mhk+b^2-k^2=0.$$

If $\tan\theta$, $\tan\theta'$ be the roots of this equation, we have

$$\tan\theta+\tan\theta'=-\frac{2hk}{a^2-h^2}\text{ and }\tan\theta\,.\,\tan\theta'=\frac{b^2-k^2}{a^2-h^2}.$$

34. Using the results of the previous example, we are to have $\theta'=\theta+90^0$;

$$\therefore\ \tan\theta'=-\cot\theta;$$

or $$\tan\theta\tan\theta'=-1;$$

$$\therefore\ \frac{b^2-k^2}{a^2-h^2}=-1,\text{ or }h^2+k^2=a^2+b^2.$$

Hence the point (h, k) is on the circle $x^2+y^2=a^2+b^2$.

35. Let the two tangents be

$$y-k=\tan\theta(x-h)\text{ and }y-k=\tan\theta'(x-h);$$

these can be combined in the one equation

$$(y-k)^2-(\tan\theta+\tan\theta')(y-k)(x-h)+\tan\theta\,.\,\tan\theta'\,.\,(x-h)^2=0.$$

Using the results of Ex. 33, this becomes

$$(a^2-h^2)(y-k)^2+2hk(y-k)(x-h)+(b^2-k^2)(x-h)^2=0.$$

Multiply this equation by a^2b^2 and we get the second result.

[*Note.* The second shape can also be obtained by the method used in Chap. VI. Ex. 35, *Aliter.*]

36. The equation to the chord of contact is

$$\frac{xh}{a^2}+\frac{yk}{b^2}=1,$$

and to the ellipse it is $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Hence the equation $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{xh}{a^2}+\frac{yk}{b^2}\right)^2$$

is satisfied where the ellipse and chord of contact meet,—that is to say, it is some locus going through the points of contact; also it is satisfied by $x=0$, $y=0$, and therefore goes through the origin.

Also since by Art. 166 $\frac{h^2}{a^2}+\frac{k^2}{b^2}>1$, it is evident that the equation stands the test of Art. 61, and therefore represents two straight lines.

37. Let (h, k) be the point of intersection of the two tangents, then, by the preceding, the equation to the radii vectores is

$$\frac{x^2}{a^4}(a^2-h^2)-\frac{2hkxy}{a^2b^2}+\frac{y^2}{b^4}(b^2-k^2)=0;$$

as these are to be at right angles, we shall have

$$\frac{a^2-h^2}{a^4}=-\frac{b^2-k^2}{b^4}\text{ (see Chap. III., Ex. 31, note)};$$

$$\therefore\quad \frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2};$$

which proves the proposition.

38. Equation to CT is $$y=\frac{k}{h}\,.\,x;$$

therefore co-ordinates of R are

$$\frac{abh}{\sqrt{(a^2k^2+b^2h^2)}}\text{ and }\frac{abk}{\sqrt{(a^2k^2+b^2h^2)}};$$

$$\therefore\quad CR^2=\frac{a^2b^2(h^2+k^2)}{a^2k^2+b^2h^2}=\frac{a^2b^2\,.\,CT^2}{a^2k^2+b^2h^2};$$

$$\therefore\quad \frac{CT^2}{CR^2}=\frac{a^2k^2+b^2h^2}{a^2b^2}.$$

39. Solving the equation to the chord of contact, namely

$$a^2yk+b^2xh=a^2b^2,$$

simultaneously with the equation to the curve, we get

$$x^2(a^2k^2+b^2h^2)-2a^2b^2hx+a^4(b^2-k^2)=0.$$

Hence $$x_1+x_2=\frac{2a^2b^2h}{a^2k^2+b^2h^2},$$

and $$x_1x_2=\frac{a^4(b^2-h^2)}{a^2k^2+b^2h^2}.$$

40. By Art. 166, we have

$$HP \,.\, HQ = (a - ex_1)(a - ex_2) = a^2 - ae(x_1 + x_2) + e^2 x_1 x_2;$$

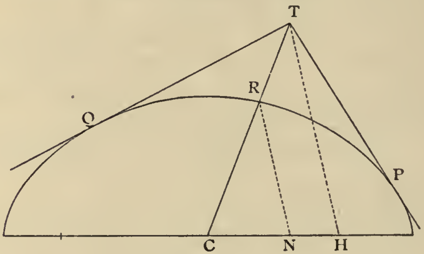
and by the preceding example this becomes

$$a^2 - \frac{2hea^3b^2}{a^2k^2 + b^2h^2} + \frac{a^4e^2(b^2 - k^2)}{a^2k^2 + b^2h^2}$$

$$= \frac{a^2b^2\{k^2 + (h - ae)^2\}}{a^2k^2 + b^2h^2}.$$

41. By similar triangles

$$CR : CT :: RN : HT;$$



$$\therefore \quad \frac{RN^2}{HT^2} = \frac{CR^2}{CT^2} = \frac{a^2b^2}{a^2k^2 + b^2h^2},$$

but $$HT^2 = k^2 + (h - ae)^2;$$

$$\therefore \quad RN^2 = \frac{a^2b^2}{a^2k^2 + b^2h^2} \,.\, HT^2$$

$$= \frac{a^2b^2\{k^2 + (h - ae)^2\}}{a^2k^2 + b^2h^2}$$

$= HP \,.\, HQ$, by preceding.

42. The excentricities being equal, if the axes of one are a, b, the axes of the other will be ma, mb, where m is some constant.

Hence the ellipses may be represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{and} \quad \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = m^2.$$

Subtracting one equation from the other, we get the equation to a straight line. Hence the ellipses will cut one another in as many points as they can be cut by this straight line, that is in two points only.

Let the third ellipse be $\frac{(x-f)^2}{a^2} + \frac{(y-g)^2}{b^2} = n^2$, and let us for shortness represent the ellipses by $U = 0$, $V = 0$, $W = 0$.

Then the common chords PP', QQ', RR' are represented by $U - V = 0$, $V - W = 0$, $W - U = 0$, and these are evidently concurrent. (See Art. 74.)

43. Let the two ellipses be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ and } \frac{x^2}{c^2}+\frac{y^2}{d^2}=1.$$

Then the equation to a common tangent, by Ex. 32, is

$$y=\pm x\sqrt{\left(\frac{d^2-b^2}{a^2-c^2}\right)}\pm\sqrt{\left(\frac{a^2d^2-b^2c^2}{a^2-c^2}\right)}.$$

Let this meet the axis major in T, and axis minor in T'; therefore CT numerically

$$=\sqrt{\left(\frac{a^2d^2-b^2c^2}{d^2-b^2}\right)},$$

and CT' numerically $$=\sqrt{\left(\frac{a^2d^2-b^2c^2}{a^2-c^2}\right)}.$$

Therefore, area of rhombus formed by the four common tangents

$$=2CT\,.\,CT'=\frac{2(a^2d^2-b^2c^2)}{\sqrt{(a^2-c^2)}\sqrt{(d^2-b^2)}}.$$

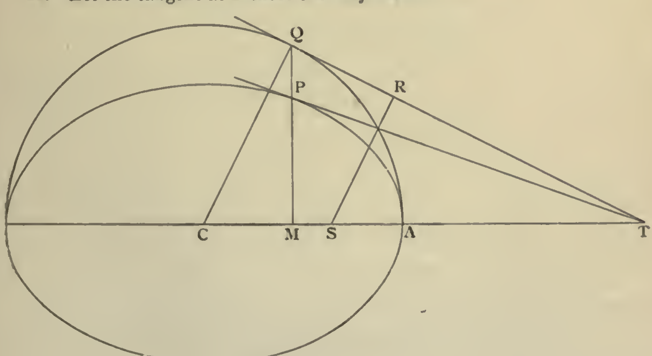
Solving the equations to the ellipses simultaneously, we find the co-ordinates of one point of intersection to be

$$\frac{ac\sqrt{(d^2-b^2)}}{\sqrt{(a^2d^2-b^2c^2)}} \text{ and } \frac{bd\sqrt{(a^2-c^2)}}{\sqrt{(a^2d^2-b^2c^2)}}.$$

Hence rectangle required $=\dfrac{4abcd\sqrt{(d^2-b^2)}\sqrt{(a^2-c^2)}}{a^2d^2-b^2c^2}$;

therefore product of areas of rectangle and rhombus $=8abcd$.

44. Let the tangent at P meet the major axis in T.



Then, by Art. 175, we have

$$CT \,.\, CM = CA^2 = CQ^2;$$

therefore CQT is a right angle. (Eucl. VI. 8, Cor.)

Hence by similar triangles

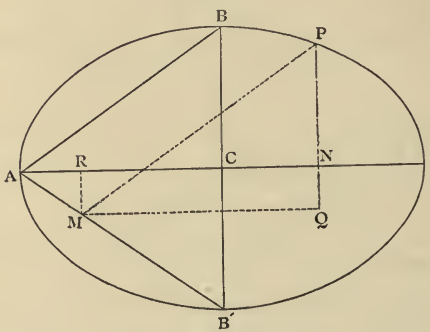
$$CQ : SR :: CT : ST;$$

$$\therefore \quad a : SR :: \frac{a^2}{x} : \frac{a^2}{x} - ae;$$

$$\therefore \quad SR = a - ex = SP.$$

45. Let (h, k) be the co-ordinates of P referred to the old axes, and (x, y) to the new.

Let $$BAC = \theta.$$



Now $$a + h = AN = AR + RN$$

$$= AR + MQ = x \cos\theta + y \cos\theta,$$

and $$k = PN = PQ - RM = y \sin\theta - x \sin\theta.$$

But $$\frac{h^2}{a^2} + \frac{k^2}{b^2} = 1;$$

$$\therefore \quad \frac{\{(x+y)\cos\theta - a\}^2}{a^2} + \frac{\{(y-x)\sin\theta\}^2}{b^2} = 1.$$

This reduces to

$$x + y)^2 + \frac{a^2}{b^2} \tan^2\theta \,.\, (y-x)^2 = 2a \sec\theta\,(x+y);$$

but $\tan\theta = \frac{b}{a}$, and consequently $\sec\theta = \frac{\sqrt{(a^2+b^2)}}{a}$.

Hence the equation becomes

$$2x^2 + 2y^2 = 2\,(x+y)\sqrt{(a^2+b^2)},$$

or $$x^2 + y^2 = (x+y)\sqrt{(a^2+b^2)}.$$

46. Let (h, k) be the point from which a tangent is drawn, then the chord of contact is $$a^2yk+b^2xh=a^2b^2,$$
or
$$y=-\frac{b^2h}{a^2k}x+\frac{b^2}{k}.$$

Also the equation to a normal is $y=mx-\dfrac{(a^2-b^2)m}{\sqrt{(b^2m^2+a^2)}}$; if these two equations are to be identical, we have
$$m=-\frac{b^2h}{a^2k}, \text{ and } \frac{b^2}{k}=-\frac{(a^2-b^2)m}{\sqrt{(b^2m^2+a^2)}}.$$

Eliminating m, we get $\dfrac{b^6}{k^2}+\dfrac{a^6}{h^2}=(a^2-b^2)^2$, which shews that (h, k) is a point on the given curve.

47. *First*, let (x_1, y_1) be any point on the ellipse; the tangent at this point is $a^2yy_1+b^2xx_1=a^2b^2$; the line from the focus perpendicular to this is
$$y=\frac{a^2y_1}{b^2x_1}(x-ae).$$

Eliminating (x_1, y_1) from these equations by the help of the condition
$$a^2y_1^2+b^2x_1^2=a^2b^2,$$
we shall have the required result. Proceed as follows:

from the first equation $\quad a^2yy_1+b^2xx_1=a^2b^2$;

from the second equation
$$a^2xy_1-b^2x_1y=a^3ey_1=a^2\sqrt{(a^2-b^2)}\,y_1.$$

Square each result and add, we get
$$(a^4y_1^2+b^4x_1^2)(x^2+y^2)=a^2\{a^4y_1^2+a^2b^4-a^2b^2y_1^2\}$$
$$=a^2\{a^4y_1^2+b^4x_1^2\};$$
therefore $x^2+y^2=a^2$ is the required locus.

Secondly, the tangent to a parabola is
$$y=mx+\frac{a}{m},$$
and the perpendicular on this from the focus is
$$y=-\frac{1}{m}(x-a).$$

From the first equation $\quad m^2x=my-a,$

and from the second equation $\quad -x=my-a.$

Subtract, and we get $(m^2+1)x=0$, or $x=0$ as the required locus.

48. The required equation is, by Art. 80,

$$b^2(x+h)^2+a^2(y+k)^2=a^2b^2,$$

or
$$b^2(x^2+2xh)+a^2(y^2+2yk)+b^2h^2+a^2k^2-a^2b^2=0.$$

But, since (h, k) is on the ellipse, we have

$$b^2h^2+a^2k^2=a^2b^2;$$

hence the equation becomes

$$b^2(x^2+2xh)+a^2(y^2+2yk)=0,$$

or
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xh}{a^2}+\frac{2yk}{b^2}=0.$$

49. Using the same axes as in the previous example, the equation to the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xh}{a^2}+\frac{2yk}{b^2}=0.$$

Now the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\left(\frac{2xh}{a^2}+\frac{2yk}{b^2}\right)(mx+ny)=0$$

is true when $mx+ny=1$ is true *simultaneously* with the equation to the ellipse: that is to say, the equation represents some locus passing through the points Q, Q'; also since it is satisfied by $x=0$, $y=0$, it goes through P. It only remains to test whether the equation represents two straight lines.

By Art. 61, the condition for this will be that

$$(nhb^2+mka^2)^2 \text{ is not } <(a^2+2nka^2)(b^2+2mhb^2),$$

or that

$$n^2h^2b^4+m^2k^2a^4-2mnhka^2b^2-2mha^2b^2-2nka^2b^2-a^2b^2 \text{ not } <0.$$

Multiply this by m^2, and the condition becomes

$$m^2n^2h^2b^4+m^4k^2a^4-2m^3nhka^2b^2-2m^3ha^2b^2-2m^2nka^2b^2-m^2a^2b^2 \text{ not } <0.$$

But if $mx+ny=1$ is to represent a chord of the ellipse, it must cut the ellipse in *real* points; to find the co-ordinates of these points we must solve $mx+ny=1$ simultaneously with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xh}{a^2}+\frac{2yk}{b^2}=0.$$

The condition that the roots of the resulting equation shall be real will be found to be identical with the condition obtained above (see Introd. § 6).

Hence the given equation does represent the two straight lines PQ, PQ'.

50. Taking the same axes and co-ordinates as in the previous example, the equation representing PQ and PQ' is to represent two lines making equal angles with the major axis, and it is therefore to be of the form $y^2 - p^2x^2 = 0$. Hence the term containing xy in the equation to PQ and PQ' is to vanish;

$$\therefore \frac{2nh}{a^2} + \frac{2mk}{b^2} = 0; \qquad \therefore \frac{m}{n} = -\frac{b^2h}{a^2k}.$$

Hence the equation to QQ', which is $mx + ny = 1$, becomes $y = \frac{b^2hx}{a^2k} + \frac{1}{n}$; or in other words, the inclination of this line to the major axis is $\tan^{-1}\frac{b^2h}{a^2k}$.

Now, with the centre as origin, the co-ordinates of P' would be $(-h, k)$, and therefore the tangent at P' would be $y = \frac{b^2hx}{a^2k} + \frac{b^2}{k}$; hence its inclination to the major axis is the same as that of QQ'.

51. Transform to the vertex A' as origin, and let $c = A'S = a(1 - e)$.

Our given equation becomes

$$y = m(x - a) + \sqrt{(m^2a^2 + a^2 - a^2e^2)},$$

or

$$y = mx - ma + ma\sqrt{\left(1 + \frac{(1+e)c}{m^2a}\right)}.$$

Expanding the surd by the Binomial Theorem we get

$$y = mx - ma + ma\left(1 + \frac{(1+e)c}{2m^2a} + \text{terms containing higher powers of } a \text{ in the denominator}\right);$$

$$\therefore y = mx + \frac{(1+e)c}{2m} + \text{\&c.}$$

When $e = 1$, and $a = \infty$ all the terms contained in the "&c." vanish, and the equation becomes $y = mx + \frac{c}{m}$.

52. Draw QN perpendicular to the major axis. Let (x, y) be the co-ordinates of Q, and (h, k) those of P.

Then $$y = QN = n \,.\, PM = nk.$$

And
$$\begin{aligned} x &= CN = CG + GN = CG + n \,.\, GM \\ &= e^2h + n(h - e^2h) \\ &= h(e^2 + n - ne^2). \end{aligned}$$

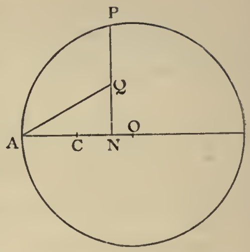
Also $$b^2h^2 + a^2k^2 = a^2b^2,$$

$$\therefore b^2\left(\frac{x}{e^2 + n - ne^2}\right)^2 + a^2\left(\frac{y}{n}\right)^2 = a^2b^2,$$

or $$\frac{x^2}{a^2(e^2 + n - ne^2)^2} + \frac{y^2}{n^2b^2} = 1.$$

Hence the locus is an ellipse, whose semi-axes are $a(e^2 + n - ne^2)$ and nb.

53. Take A as origin, and let the radius of the circle be a.



Let the co-ordinates of Q be (x, y) and those of P be (x, k).

Now $$AQ^2 = PN^2,$$
$$\therefore\ x^2 + y^2 = k^2.$$

But $x^2 + k^2 - 2ax = 0$, since P is on the circle; (Art. 88, III.).

Add these two equations together, and we get $2x^2 + y^2 - 2ax = 0$, which is the locus of Q.

Writing this in the shape $\dfrac{(x - \frac{1}{2}a)^2}{\frac{1}{4}a^2} + \dfrac{y^2}{\frac{1}{2}a^2} = 1$, we see that the locus is an ellipse with its centre C at the middle point of AO, and its major axis perpendicular to AO.

If H be one focus, $CH^2 = (\text{semi-major axis})^2 - (\text{semi-minor axis})^2$

$$= \tfrac{1}{2}a^2 - \tfrac{1}{4}a^2 = \tfrac{1}{4}a^2;\ \therefore\ CH = \pm\frac{a}{2},$$

or in other words the foci are at a distance of half the radius, above and below C.

54. The equation to CP is $y = \dfrac{k}{h}\,.\,x$, if (h, k) be the co-ordinates of P.

The normal at P is $y - k = \dfrac{a^2k}{b^2h}(x - h)$;

$$\therefore\ \tan CPG = \frac{\dfrac{a^2k}{b^2h} - \dfrac{k}{h}}{1 + \dfrac{a^2k^2}{b^2h^2}} = \frac{(a^2 - b^2)hk}{a^2k^2 + b^2h^2}$$

$$= \frac{a^2e^2hk}{a^2b^2} = \frac{e^2hk}{b^2}.$$

55. In the previous example put $k = b\sin\phi$, $h = a\cos\phi$;

$$\therefore\ \tan CPG = \frac{e^2ab\sin\phi\,.\,\cos\phi}{b^2}$$

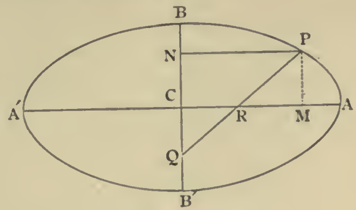
$$= \frac{e^2a}{2b}\sin 2\phi;$$

the greatest value of this is when $\sin 2\phi = 1$.

In this case $\tan CPG = \dfrac{e^2a}{2b}$.

56. Let the semi-axes be $2b$ and b.

Let the co-ordinates of P be (x, y).



Then
$$4b^2y^2+b^2x^2=4b^4,$$
or
$$4y^2+x^2=4b^2.$$
But
$$QN^2+PN^2=QP^2=4b^2 \text{ by hypothesis;}$$
$$\therefore\ QN^2+PN^2=4y^2+x^2.$$
But
$$PN^2=x^2,$$
$$\therefore\ QN^2=4y^2,$$
$$\therefore\ QN=2y=2CN;$$
$$\therefore\ QP=2PR, \text{ or } R \text{ is middle point of } QP.$$

[*Note.* More generally, whatever be the axes of the ellipse, if QP be equal to the semi-major axis, then PR is equal to the semi-minor axis.

For let the equation to the ellipse be
$$a^2y^2+b^2x^2=a^2b^2;$$
or
$$\frac{a^2y^2}{b^2}+x^2=a^2.$$
Now
$$QN^2+PN^2=PQ^2=a^2, \text{ by hypothesis,}$$
or
$$QN^2+x^2=a^2;$$
$$\therefore\ QN^2=\frac{a^2y^2}{b^2}, \text{ or } QN=\frac{ay}{b};$$
$$\therefore\ QN : PM :: a : b.$$
But, by similar triangles,
$$QN : PM :: PQ : PR;$$
or
$$QN : PM :: a : PR;$$
$$\therefore\ PR=b.$$

This property affords a method of describing the ellipse mechanically by an instrument called the trammel.

Let a straight rod PQ of length a carry a pencil point at P; mark off $PR=b$, and at Q and R let two pegs project from the surface of the rod. Then if these two pegs be constrained to move in two grooves CA, CB, at right angles to each other, the pencil P will trace out an ellipse.]

57. In the figure to Art. 175, let O be the centre of the inscribed circle of SPH: then since PG bisects the angle SPH, it is evident that O lies on PG.

Draw ON perpendicular to the major axis.

Let (x, y) be the co-ordinates of O, and (h, k) of P.

By Trigonometry, since ON is the radius of the inscribed circle,

$$ON = \frac{\text{area of } SPH}{\text{semi-perimeter of } SPH},$$

$$\therefore\ y = \frac{aek}{a+ae} = \frac{ek}{1+e}.$$

Also $$GN : GM :: ON : PM :: e : 1+e,$$

$$\therefore\ GN = \frac{e}{1+e} \cdot GM = \frac{e}{1+e}(h - e^2h) = eh(1-e),$$

$$\therefore\ x = CG + GN = e^2h + eh(1-e) = eh.$$

But $\dfrac{h^2}{a^2} + \dfrac{k^2}{b^2} = 1$, since P is on the ellipse;

$$\therefore\ \frac{x^2}{a^2e^2} + \frac{y^2}{\dfrac{b^2e^2}{(1+e)^2}} = 1;$$

this is evidently an ellipse with semi-axes ae and $\dfrac{be}{1+e}$.

58. Using the figure to Art. 175, let SZ cut PG in Q, and let HZ' cut PG in Q'.

Then, by similar triangles,

$$\begin{aligned} GQ : HZ :: SG : SH &:: ae + e^2x : 2ae \\ &:: a + ex : 2a \\ &:: SP : AA', \end{aligned}$$

$$\therefore\ GQ \,.\, AA' = SP \,.\, HZ.$$

Similarly, $$GQ' \,.\, AA' = HP \,.\, SZ'.$$

But, by similar triangles HZP, $SZ'P$, we have

$$HZ : HP :: SZ' : SP;$$

$$\therefore\ SP \,.\, HZ = HP \,.\, SZ';$$

$\therefore\ GQ = GQ'$, or Q and Q' are coincident.

59. The equations to the two lines joining (h, k) to the foci are respectively $hy - kx + aey - aek = 0$, and $hy - kx - aey + aek = 0$; and these can be combined in the one equation

$$(hy - kx)^2 - a^2e^2(y-k)^2 = 0.$$

60. Transforming to axes through (h, k) parallel to the old axes,—which is done by writing $x+h$ for x and $y+k$ for y,—the equation of Ex. 35 becomes

$$(a^2-h^2)y^2+2xyhk+(b^2-k^2)x^2=0,$$

and the equation in Ex. 59 becomes

$$(hy-kx)^2-a^2e^2y^2=0;$$

or

$$(h^2-a^2e^2)y^2-2xyhk+k^2x^2=0.$$

By Ex. 12, Chap. VII. the condition that both pairs of lines should have the same bisectors is

$$\frac{2hk}{a^2-h^2-b^2+k^2}=\frac{-2hk}{h^2-a^2e^2-k^2},$$

which is easily seen to be satisfied.

[*Note.* This example proves the last result of Art. 186.]

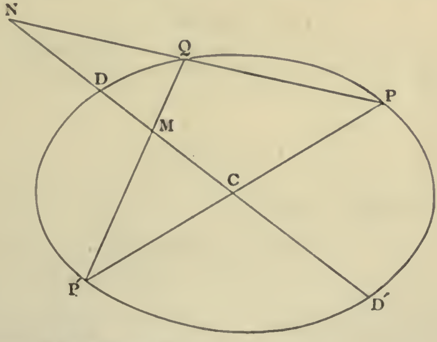
CHAPTER X.

1. The co-ordinates of D are $\left(-\frac{ay'}{b}, \frac{bx'}{a}\right)$: hence the equation to PD is

$$y-y'=\frac{\frac{bx'}{a}-y'}{-\frac{ay'}{b}-x'}(x-x'),$$

which reduces to $\qquad ay(ay'+bx')+bx(bx'-ay')=a^2b^2.$

2. Take CP, CD as axes: let Q be the given point, and (h, k) its co-ordinates.



The equation to the ellipse is

$$b'^2x^2+a'^2y^2=a'^2b'^2.$$

Now the equation to QP is $y=\frac{k}{h-a'}(x-a')$.

Put $x=0$, $\therefore\ CN=\frac{ka'}{-h+a'}$.

Similarly $CM=\frac{ka'}{h+a'}$;

$$\therefore\ CN\,.\,CM=\frac{k^2a'^2}{a'^2-h^2}=\frac{k^2a'^2b'^2}{a'^2b'^2-b'^2h^2}=\frac{k^2a'^2b'^2}{a'^2k^2}=b'^2=CD^2.$$

3. Let the co-ordinates of P be (x, y) and of P' be (h, k);

$\therefore$ co-ordinates of D are $\left(-\frac{ay}{b}, \frac{bx}{a}\right)$ and of D' they are $\left(-\frac{ak}{b}, \frac{bh}{a}\right)$.

Hence, by Art. 11, area of $PCP'=\pm\frac{1}{2}(xk-yh)$;

also, area of $DCD'=\pm\frac{1}{2}\left\{-\frac{ak}{b}\cdot\frac{bx}{a}+\frac{ay}{b}\cdot\frac{bh}{a}\right\}=\pm\frac{1}{2}\{xk-yh\}$;

$\therefore$ area of PCP' = area of DCD'.

4. Let (h, k) be the co-ordinates of P; then the normal at P is

$$y-k=\frac{a^2k}{b^2h}(x-h),$$

or
$$b^2hy-b^2hk=a^2kx-a^2hk.$$

Similarly the normal at D is $abky-b^2hk=-abhx-a^2hk$.

Subtract this last equation from the previous, and we get

$$(b^2h-abk)\,y=(a^2k+abh)\,x.$$

Since this equation is derived from the equations to the two normals it passes through their intersection K; and since it is satisfied by $x=0$, $y=0$, it goes through C.

Hence it is the line CK.

Writing it in the shape $y=\frac{a^2k+abh}{b^2h-abk}\cdot x$,

we see that it is perpendicular to

$$y=-\frac{b^2h-abk}{a^2k+abh}x+\frac{a^2b^2}{a^2k+abh},$$

which by Ex. 1 is the equation to PD.

5. See figure to Art. 175.

By Art. 194, if p be the perpendicular from C on the tangent, we have

$$p\,.\,CD=ab.$$

Also, by Art. 177, $PG^2 = \frac{b^2}{a^2} \cdot rr' = \frac{b^2}{a^2} \cdot CD^2$, by Art. 193,

$$\therefore\ PG = \frac{b}{a} \cdot CD.$$

Similarly
$$PG' = \frac{a}{b} \cdot CD;$$

$$\therefore\ GG' = \frac{a^2 - b^2}{ab} \cdot CD = \frac{a^2 - b^2}{ab} \cdot \frac{ab}{p} = \frac{a^2 - b^2}{p};$$

$$\therefore\ p \cdot GG' = a^2 - b^2.$$

6. The equation to a tangent is $y = mx + \sqrt{(a^2m^2 + b^2)}$, and the equation to the perpendicular on it is $\quad y = -\frac{1}{m}x.$

Eliminating m between these, we get

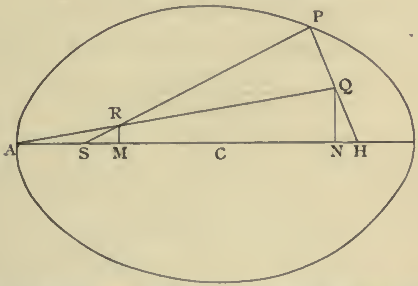
$$(x^2 + y^2)^2 = a^2x^2 + b^2y^2,$$

or
$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta.$$

7. CQ is evidently parallel to

$$SP \text{ and } = \tfrac{1}{2}SP;$$

$$\therefore\ CQ + QH = \tfrac{1}{2}(SP + PH) = AC;$$



$\therefore$ locus of Q is an ellipse with foci C, H, and axes half those of the given ellipse. The equation to this ellipse with the middle point of CH as origin will evidently be $\frac{x^2}{\frac{1}{4}a^2} + \frac{y^2}{\frac{1}{4}b^2} = 1$ and therefore the equation referred to A as origin will be $\quad \frac{(x - a - \frac{1}{2}ae)^2}{\frac{1}{4}a^2} + \frac{y^2}{\frac{1}{4}b^2} = 1.$

If therefore (h, k) are the co-ordinates of Q referred to A as origin,

$$\frac{(h - a - \frac{1}{2}ae)^2}{\frac{1}{4}a^2} + \frac{k^2}{\frac{1}{4}b^2} = 1.$$

Let the co-ordinates of R be (x_1, y_1), with A as origin.

Then, by similar triangles, $QN : RM :: AQ : AR$

$$:: AC : AS$$

$$:: a : a - ae;$$

$$\therefore\ k = \frac{y_1}{1-e}.$$

Also $$AN : AM :: AQ : AR$$

$$:: a : a - ae;$$

$$\therefore\ h = \frac{x_1}{1-e};$$

$$\therefore\ \frac{\left(\frac{x_1}{1-e} - a - \frac{1}{2}ae\right)^2}{\frac{1}{4}a^2} + \frac{\left(\frac{y_1}{1-e}\right)^2}{\frac{1}{4}b^2} = 1.$$

Hence the locus of R is an ellipse.

8. The equation to the ellipse referred to the vertex as origin is

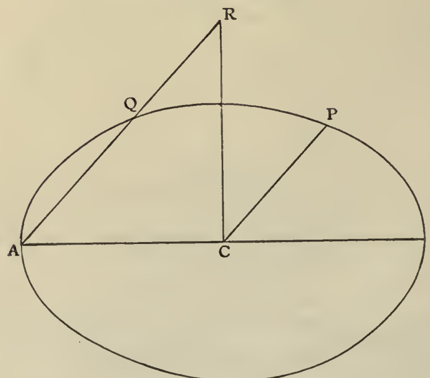
$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1;$$

or $$a^2y^2 + b^2x^2 - 2ab^2x = 0.$$

By Art. 8, this becomes $r(a^2 \sin^2\theta + b^2 \cos^2\theta) = 2ab^2 \cos\theta$.

9. By the previous example, if the angle $QAC = \theta$, we have

$$AQ = \frac{2ab^2 \cos\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta}.$$



Also $AR = a \sec\theta$;

$$\therefore AQ \,.\, AR = \frac{2a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}.$$

But, by Art. 206, $$CP^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta};$$

$$\therefore AQ \,.\, AR = 2CP^2.$$

10. Let the angle $PAA' = \theta$; then since APA' is a right angle,

$$AP = 2a\cos\theta.$$

Also, by Ex. 8, $$AQ = \frac{2ab^2\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta};$$

$$\therefore \frac{AP}{AQ} = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{b^2}.$$

Similarly $$\frac{A'P}{A'Q'} = \frac{a^2\cos^2\theta + b^2\sin^2\theta}{b^2};$$

$$\therefore \frac{AP}{AQ} + \frac{A'P}{A'Q'} = \frac{a^2+b^2}{b^2}.$$

11. Let Q be the intersection of two circles, one on CP as diameter, and the other on CD.

If the excentric angle of P be ϕ, the co-ordinates of P are $(a\cos\phi,\ b\sin\phi)$, and of D they are $(-a\sin\phi,\ b\cos\phi)$.

The equation to the circle on CP as diameter is

$$x^2 + y^2 = a\cos\phi \,.\, x + b\sin\phi \,.\, y.$$

Similarly the equation to the circle on CD as diameter is

$$x^2 + y^2 = -a\sin\phi \,.\, x + b\cos\phi \,.\, y.$$

Eliminate ϕ by squaring these equations and adding;

$$\therefore 2(x^2+y^2)^2 = a^2x^2 + b^2y^2.$$

This is therefore the locus of Q.

12. It is evident that the point named in this example is the point Q of the previous example, hence the result has been already obtained.

13. From the two given equations we get

$$r + re\cos\theta = r\sin\theta + r\cos\theta + re\cos\theta,$$

or $$\sin\theta + \cos\theta = 1.$$

Hence we shall find $\theta = 0^0$ or 90^0.

Putting these values into either equation we get the values $r = a(1-e)$ and $r = a(1-e^2)$ respectively.

14. See the figure to Art. 162.

Let S be the pole, and SH the initial line.

Let RSR' be the latus rectum through S. Now with S as origin the rectangular co-ordinates of R are $(0, l)$: of R' they are $(0, -l)$: of L they are $(2ae, l)$: of L' they are $(2ae, -l)$: of B they are (ae, b): of B' they are $(ae, -b)$.

The general form of the tangent with S as origin is

$$a^2yy' + b^2(x-ae)(x'-ae) = a^2b^2.$$

Hence the tangent at R is

$$a^2yl + b^2(x-ae)(-ae) = a^2b^2,$$

which reduces to $$y - ex = l;$$

and in polar co-ordinates this becomes $r(\sin\theta - e\cos\theta) = l$.

Similarly the tangent at R' is $r(\sin\theta + e\cos\theta) = -l$.

The tangent at L is $r(\sin\theta + e\cos\theta) = a(1+e^2)$.

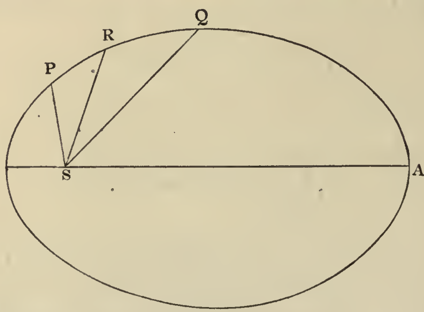
The tangent at L' is $r(\sin\theta - e\cos\theta) = -a(1+e^2)$.

The tangent at B is $y = b$, or $r\sin\theta = b$.

The tangent at B' is $y = -b$, or $r\sin\theta = -b$.

Note. The results in the answers are obtained by using SA' as initial line.

15. Using the notation of Art. 205, it is evident that in this case α is constant.



Now the tangent at P is, by Art. 205,

$$r\{e\cos\theta + \cos(\alpha-\beta-\theta)\} = l.$$

And tangent at Q is $r\{e\cos\theta + \cos(\alpha+\beta-\theta)\} = l$.

These equations evidently coincide when $\theta = \alpha$, or in other words, the tangents intersect on the *fixed* line $\theta = \alpha$, which is evidently the line SR bisecting the angle PSQ.

16. If $SP=\dfrac{a(1-e^2)}{1+e\cos\theta}$, it follows by putting $180^0+\theta$ for θ that

$$Sp=\frac{a(1-e^2)}{1-e\cos\theta}.$$

Hence $$SQ^2=\frac{a^2(1-e^2)^2}{1-e^2\cos^2\theta}=\frac{a^2(1-e^2)^2}{\sin^2\theta+(1-e^2)\cos^2\theta}$$

$$=\frac{a^4(1-e^2)^2}{a^2\sin^2\theta+a^2(1-e^2)\cos^2\theta}=\frac{b^4}{a^2\sin^2\theta+b^2\cos^2\theta}.$$

Comparing this result with that of Art. 206, we see that the locus of Q is an ellipse, whose semi-axes are b and $\dfrac{b^2}{a}$; hence the required excentricity is

$$\sqrt{1-\frac{b^4}{a^2b^2}}=\sqrt{1-\frac{b^2}{a^2}}=e.$$

17. Let the polar equations to the two ellipses be

$$r=\frac{a(1-e_1^2)}{1+e_1\cos\theta}, \text{ and } r=\frac{a(1-e_2^2)}{1+e_2\cos\theta}.$$

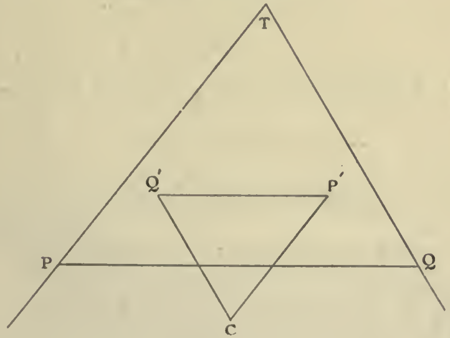
Solving these simultaneously we get $r=a(1+e_1e_2)$ and $\cos\theta=-\dfrac{e_1+e_2}{1+e_1e_2}$.

18. By Art. 208, we know that the ratio of the two tangents is the same as the ratio of the diameters parallel to them; hence the question becomes, "find the greatest and least values of $\dfrac{CP}{CD}$, where CP and CD are semi-diameters."

It is obvious that the greatest value of a semi-diameter is a, and the least value is b; hence the ratio lies between the limits $\dfrac{b}{a}$ and $\dfrac{a}{b}$.

19. Now the angle at C is equal to the angle at T, and by Art. 208,

$$TP : TQ :: CP' : CQ'.$$



Hence the triangles TPQ, $CP'Q'$ are similar.

$\therefore$ angle $P'Q'C$ = angle TQP, from which it is easily seen that PQ is parallel to $P'Q'$.

20. Let the line from O make an angle α with the major axis;

$\therefore$ by Art. 208, we have $\quad OP \cdot Op = \dfrac{a^2k^2 + b^2h^2 - a^2b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$.

Also, by Art. 206, we have $\quad CD^2 = \dfrac{a^2b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$;

$$\therefore \frac{OP \cdot Op}{CD^2} = \frac{a^2k^2 + b^2h^2 - a^2b^2}{a^2b^2} = \frac{k^2}{b^2} + \frac{h^2}{a^2} - 1.$$

21. See figure to Art. 175.

If the angle HPZ is minimum, its sine is a minimum; hence $\dfrac{HZ^2}{HP^2}$ is to be a minimum. But $\dfrac{HZ^2}{HP^2} = \dfrac{b^2}{2ar - r^2}$, by Art. 181.

$\therefore \dfrac{HZ^2}{HP^2} = \dfrac{b^2}{a^2 - (a-r)^2}$, and this is evidently a minimum when $r = a$.

22. Using the notation of Art. 196, we shall require the value of CP which will make $\dfrac{p}{CP}$ a minimum.

But $$\frac{p^2}{CP^2} = \frac{a^2b^2}{(a^2+b^2)\,CP^2 - CP^4} = \frac{a^2b^2}{\left(\dfrac{a^2+b^2}{2}\right)^2 - \left(\dfrac{a^2+b^2}{2} - CP^2\right)^2},$$

and this is evidently a minimum when $CP^2 = \dfrac{a^2+b^2}{2}$.

23. Let CT and Ct meet the curve in Q and D respectively.

Take CP and its conjugate as axes of y and x, and let the co-ordinates of Q be (h, k).

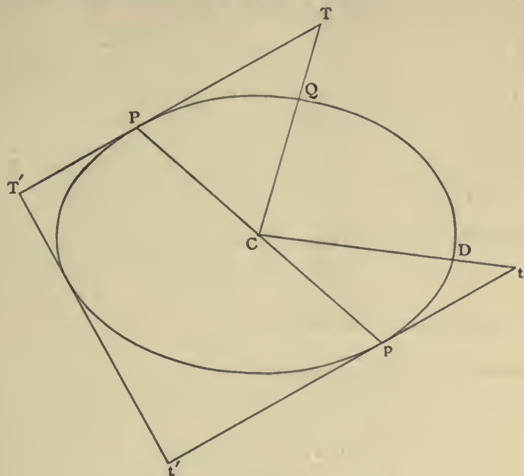
$$\therefore \text{ equation to } CQ \text{ is } y = \frac{k}{h}x;$$

$$\text{equation to } CD \text{ is } y = -\frac{b'^2h}{a'^2k}x;$$

$$\text{equation to } PT \text{ is } y = b'; \text{ hence } PT = \frac{hb'}{k}.$$

Also $pt = \dfrac{a'^2k}{b'h}$; $\quad \therefore PT \cdot pt = a'^2$.

Equation to $T't'$ can be taken as $a'^2yy' + b'^2xx' = a'^2b'^2$;



Hence $$PT' = \frac{a'(a'b' - a'y')}{b'x'} \text{ and } pt' = \frac{a'(a'b' + a'y')}{b'x'};$$

$$\therefore\ PT' \cdot pt' = a'^2 = PT \cdot pt; \quad \therefore\ PT : PT' :: pt' : pt.$$

24. Let T be the point at which the tangent is drawn. Take CP and CD as axes, and let the equation to Ct be $y = nx$.

Let the equation to Tt be $y = mx + \sqrt{(a_1^2m^2 + b_1^2)}$;

then the equation to Pp is $\quad y = m(x - a_1)$,

and to Dd is $\quad y - b_1 = mx$.

Hence $$Ct^2 = \frac{(a_1^2m^2 + b_1^2)(1 + n^2)}{(n - m)^2}; \quad Cp^2 = \frac{a_1^2m^2(1 + n^2)}{(n - m)^2};$$

$$Cd^2 = \frac{b_1^2(1 + n^2)}{(n - m)^2}.$$

$$\therefore\ Cp^2 + Cd^2 = Ct^2.$$

25. At the point P let the tangent PQ be drawn, equal to n times the semi-conjugate CD.

Let co-ordinates of Q be (h, k) and those of P be (x', y').

Now $$PQ^2 = n^2 \cdot CD^2.$$

$$\therefore\ (x' - h)^2 + (y' - k)^2 = n^2\left(\frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2}\right).$$

Also $$a^2ky' + b^2hx' = a^2b^2.$$
And $$a^2y'^2 + b^2x'^2 = a^2b^2.$$

From the last two equations we get $x' - h = -\dfrac{a^2y'}{b^2x'}(y' - k)$, and substituting this in the first equation we get $y' - k = \dfrac{nbx'}{a}$.

Similarly $$x' - h = -\frac{nay'}{b}.$$

From these two we get $ay' = \dfrac{nbh + ak}{n^2+1}$; $bx' = \dfrac{bh - nak}{n^2+1}$;
and substituting these in the third equation, we get

$$\frac{h^2}{a^2(n^2+1)} + \frac{k^2}{b^2(n^2+1)} = 1.$$

26. The equation to one tangent being

$$y = mx + \sqrt{(a^2m^2 + b^2)},$$

the equation to the other is by Art. 188

$$y = -\frac{b^2x}{a^2m} + \sqrt{\left(\frac{b^4}{a^2m^2} + b^2\right)},$$

or $$\frac{amy}{b} = -\frac{bx}{a} + \sqrt{(a^2m^2 + b^2)}.$$

Subtracting the first equation from the last one, we get

$$\frac{amy}{b} - y = -\frac{bx}{a} - mx,$$

$$\therefore\ m = \frac{b(ay - bx)}{a(ay + bx)}.$$

Substitute this value in the first equation, and clear of surds, and we obtain the required locus.

27. By Art. 187, we have

$$r^2(a^2\sin^2\alpha + b^2\cos^2\alpha) + 2r(a^2y'\sin\alpha + b^2x'\cos\alpha) = 0,$$

since (x', y') is on the ellipse.

Also $$\tan\phi = -\frac{b^2x'}{a^2y'}.$$

Hence $$r = -\frac{2(a^2y'\sin\alpha + b^2x'\cos\alpha)}{a^2\sin^2\alpha + b^2\cos^2\alpha}$$

$$= -\frac{2a^2y'(\sin\alpha - \tan\phi\,.\cos\alpha)}{a^2\sin^2\alpha + b^2\cos^2\alpha};$$

$$\therefore\ r \propto y'(\sin\alpha - \tan\phi\cos\alpha),$$

or $$\propto \frac{y'\sin(\alpha - \phi)}{\cos\phi}.$$

28. The angle α in this question is equivalent to $\alpha-\phi$ in the previous question:
$$\therefore\ PQ \operatorname{cosec} \alpha \propto \frac{y'}{\cos\phi}.$$

Similarly
$$PR \operatorname{cosec} \beta \propto \frac{y'}{\cos\phi};$$

$\therefore$ $PQ \operatorname{cosec} \alpha : PR \operatorname{cosec} \beta$ is constant.

29. The tangents at the ends of the latus rectum of the parabola meet on the axis of the parabola; hence by Art. 201, the axis of the parabola will be that diameter of the ellipse which bisects the latus rectum; but as it bisects it at right angles, this diameter must be one of the two axes of the ellipse.

Hence the problem becomes simply this: "In a given ellipse, find the double ordinate or double abscissa of the point at which the tangent makes 45^0 with the axis." Let co-ordinates of the point be (h, k).

Here
$$m' = \pm 1,\ \therefore\ a^2k = \pm b^2h;$$
also
$$a^2k^2 + b^2h^2 = a^2b^2;$$
hence we get
$$2h = \frac{2a^2}{\sqrt{(a^2+b^2)}};\ 2k = \frac{2b^2}{\sqrt{(a^2+b^2)}}.$$

30. Let the ends of the diameters be P and Q, and let CR be perpendicular to PQ.

Let co-ordinates of P be (h, k) and of Q be (x', y'). Then since CP is perpendicular to CQ, we have
$$\frac{k}{h}\cdot\frac{y'}{x'} = -1, \text{ or } \frac{k}{h} = -\frac{x'}{y'};$$
$$\therefore\ \frac{x'^2+y'^2}{h^2+k^2} = \frac{y'^2}{h^2} = \frac{y'^2\left(a^2+\dfrac{b^2x'^2}{y'^2}\right)}{h^2\left(a^2+\dfrac{b^2k^2}{h^2}\right)} = \frac{a^2b^2}{a^2h^2+b^2k^2}$$
$$= \frac{a^2k^2+b^2h^2}{a^2h^2+b^2k^2}.$$

Now since the area of PCQ is $\frac{1}{2}\,CP\,.\,CQ$, and also is $\frac{1}{2}\,CR\,.\,PQ$, we have
$$CR^2 = \frac{CP^2\,.\,CQ^2}{PQ^2} = \frac{(x'^2+y'^2)\,(h^2+k^2)}{x'^2+y'^2+h^2+k^2}$$
$$= \frac{h^2+k^2}{1+\dfrac{h^2+k^2}{x'^2+y'^2}} = \frac{h^2+k^2}{1+\dfrac{a^2h^2+b^2k^2}{a^2k^2+b^2h^2}}$$
$$= \frac{a^2b^2\,(h^2+k^2)}{a^2k^2+b^2h^2+a^2h^2+b^2k^2} = \frac{a^2b^2}{a^2+b^2};$$
a constant quantity.

This result is often useful in the form
$$\frac{1}{CR^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

31. Let AP be a chord through A; Q its middle point, and QN the ordinate of Q.

As in Art. 188, we have $y' = -\frac{b^2}{a^2}\cot\theta \, . \, x'$; where (x', y') are the co-ordinates of Q.

But
$$\cot\theta = -\frac{NA}{QN} = -\frac{a-x'}{y'}.$$

Hence
$$y' = \frac{b^2}{a^2} \cdot \frac{a-x'}{y'} \, . \, x',$$

or
$$a^2y'^2 + b^2x'^2 - ab^2x' = 0.$$

Hence the locus is an ellipse, passing through the centre and end of axis of the given ellipse, and with excentricity equal to that of the given ellipse.

32. Let (h, k) be the fixed point, and (x', y') the middle point of the chord, so that the equation to the chord is $y - y' = \frac{k-y'}{h-x'}(x - x')$.

As in Art. 188, we have $y' = -\frac{b^2}{a^2}\cot\theta \, . \, x'$;

but
$$\cot\theta = \frac{h-x'}{k-y'};$$

$\therefore$ the equation is
$$y' = -\frac{b^2x'}{a^2} \cdot \frac{h-x'}{k-y'},$$

or $a^2y'^2 + b^2x'^2 - a^2ky' - b^2hx' = 0$; an ellipse of excentricity equal to that of the given one.

33. Using the figure to Art. 205, let us suppose that PSP' is a right angle, and that PS, $P'S$, meet the curve again in Q' and Q''.

Let $A'SP = \theta$, and therefore $A'SP' = 90^0 + \theta$.

$$\therefore \; SP = \frac{l}{1 + e\cos\theta}, \quad SQ' = \frac{l}{1 - e\cos\theta};$$

$$\therefore \; PQ' = \frac{2l}{1 - e^2\cos^2\theta}.$$

Similarly,
$$P'Q'' = \frac{2l}{1 - e^2\sin^2\theta};$$

$$\therefore \; PQ' \, . \, P'Q'' = \frac{4l^2}{1 - e^2 + \frac{e^4}{4}\sin^2 2\theta}.$$

This is a maximum when $\theta = 0^0$, and a minimum when $\theta = 45^0$.

34. As in the preceding, we get $SP \, . \, SP' = \dfrac{l^2}{1 - e^2 \cos^2 \theta}$,

and
$$SQ \, . \, SQ' = \frac{l^2}{1 - e^2 \sin^2 \theta};$$

$$\therefore \frac{1-e^2}{SP \, . \, SP'} + \frac{1-e^2}{SQ \, . \, SQ'} = \frac{1-e^2}{l^2} \left\{1 - e^2 \cos^2 \theta + 1 - e^2 \sin^2 \theta\right\}$$

$$= \frac{1}{a^2(1-e^2)}(1 + 1 - e^2) = \frac{1}{a^2(1-e^2)} + \frac{1}{a^2} = \frac{1}{b^2} + \frac{1}{a^2}.$$

35. Let $A'SP = \alpha - \beta$, and $A'SQ = \alpha + \beta$.

$\therefore$ by § 205 the equation to PQ is $r = \dfrac{l}{e \cos \theta + \sec \beta \, . \cos(\alpha - \theta)}$.

Similarly the equation to pq is $r = \dfrac{l}{e \cos \theta - \sec \beta \, . \cos(\alpha - \theta)}$.

Where these intersect we have $\alpha - \theta = 90^0$,

or
$$\theta = \alpha - 90;$$

$\therefore$ $A'ST = \alpha - 90^0$, or ST bisects the angle PSq.

Also for the point of intersection the above equations become

$$r = \frac{l}{e \cos \theta} = \frac{l}{e \sin \alpha};$$

$\therefore$ $r \cos A'ST = \dfrac{l}{e}$ = perpendicular from S on directrix, by Art. 161.

$\therefore$ T is on the directrix.

36. Using the figure of Art. 175, we have

$$\sin^2 HPZ = \sin HPZ \, . \sin SPZ' = \frac{pp'}{rr'} = \frac{b^2}{r(2a - r)}; \quad \text{(Art. 181.)}$$

$$\therefore \tan HPZ = \frac{b}{\sqrt{(2ar - r^2 - b^2)}}.$$

Substitute the value of r from Art. 204, and we at once get the second form.

37. Let the co-ordinates of P be (h, k), and those of D be $\left(-\dfrac{ak}{b}, \dfrac{bh}{a}\right)$.

The length of perpendicular from P on $y = x \tan \alpha$ is $k \cos \alpha - h \sin \alpha$, by Art. 47.

Similarly the perpendicular from D is $\dfrac{bh}{a} \cos \alpha + \dfrac{ak}{b} \sin \alpha$; the sum of the squares of these perpendiculars is

$$\left(\frac{b^2h^2}{a^2} + k^2\right) \cos^2\alpha + \left(\frac{a^2k^2}{b^2} + h^2\right) \sin^2\alpha,$$

or $b^2 \cos^2\alpha + a^2 \sin^2\alpha$; since (h, k) is on the ellipse.

38. Let the tangent at P meet the two tangents at the ends of the diameter CQ in the points K and L; and let CR be perpendicular to KL.

Then the area of the parallelogram is $2CR.KL$.

Now the four tangents are evidently

$$\left.\begin{aligned} ay\sin\phi + bx\cos\phi &= ab;\\ ay\sin\phi + bx\cos\phi &= -ab;\\ ay\sin\phi' + bx\cos\phi' &= ab;\\ ay\sin\phi' + bx\cos\phi' &= -ab. \end{aligned}\right\}$$

Hence the co-ordinates of K will be

$$\frac{a(\sin\phi' - \sin\phi)}{\sin(\phi' - \phi)},\ \frac{b(\cos\phi' - \cos\phi)}{\sin(\phi' - \phi)}.$$

Similarly the co-ordinates of L will be

$$\frac{a(\sin\phi' + \sin\phi)}{\sin(\phi' - \phi)},\ \frac{b(\cos\phi' + \cos\phi)}{\sin(\phi' - \phi)}.$$

Hence $$KL = \frac{2\sqrt{(a^2\sin^2\phi + b^2\cos^2\phi)}}{\sin(\phi' - \phi)}.$$

Also since CR is perpendicular to $ay\sin\phi + bx\cos\phi = ab$, its length is

$$\frac{ab}{\sqrt{(a^2\sin^2\phi + b^2\cos^2\phi)}};$$

$\therefore$ area of parallelogram $= \dfrac{4ab}{\sin(\phi' - \phi)}$.

This area is least when $\phi' - \phi$ is 90^0, which by Art. 197 is the case of conjugate diameters.

39. By Art. 187 if (x, y) be the co-ordinates of the middle point named, we have the equation

$$c^2(a^2\sin^2\theta + b^2\cos^2\theta) + 2c(a^2y\sin\theta + b^2x\cos\theta) + a^2y^2 + b^2x^2 - a^2b^2 = 0,$$

where θ is determined by the equation $y = -\dfrac{b^2}{a^2}\cot\theta . x$. (see Art. 188.)

Substitute this value of θ in the first equation, and we get

$$c^2 . \frac{a^2y^2 + b^2x^2}{a^4y^2 + b^4x^2} + \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

40. Let the chord OE make an angle θ with the axis of x; and let r_1, r_2 be the lengths of the two portions of this chord measured from O to the curve.

Then $OE = \frac{1}{2}(r_1 + r_2)$, where r_1 and r_2 are the roots of the equation of Art. 187.

$$\therefore\ OE^2 = \tfrac{1}{4}(r_1 + r_2)^2;$$

$$\therefore\ OE^2 = \frac{(a^2k\sin\theta + b^2h\cos\theta)^2}{(a^2\sin^2\theta + b^2\cos^2\theta)^2}. \qquad \text{(Introd. § II.)}$$

But by Art. 206 $$CP^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta};$$

$$\therefore \frac{OE^2}{CP^4} = \frac{(a^2k\sin\theta + b^2h\cos\theta)^2}{a^4b^4}.$$

By altering θ into $90^0 + \theta$ we have $\dfrac{OF^2}{CQ^4} = \dfrac{(a^2k\cos\theta - b^2h\sin\theta)^2}{a^4b^4}$. Hence the result at once follows.

41. [See figure to Art. 192.] The equation to the tangent at P is

$$a^2yy' + b^2xx' = a^2b^2.$$

The equation to the tangent at D is $a^2y\dfrac{bx'}{a} - b^2x\dfrac{ay'}{b} = a^2b^2$,

or $$x'y - y'x = ab.$$

Solving the two equations simultaneously we get for the required co-ordinates $$\frac{bx' - ay'}{b}, \quad \frac{bx' + ay'}{a}.$$

42. Let (h, k) be the co-ordinates of the point where the tangents at P and D meet; then by the previous example

$$h = \frac{bx' - ay'}{b}, \quad k = \frac{bx' + ay'}{a}.$$

Substituting these values in the latter equation of Ex. 35, Chap. IX. we get

$$(2b^2x'^2 + 2a^2y'^2 - a^2b^2)(a^2y^2 + b^2x^2 - a^2b^2) = (abx'y + a^2yy' + b^2xx' - abxy' - a^2b^2)^2,$$

or $$a^2b^2(a^2y^2 + b^2x^2 - a^2b^2) = \{bx(bx' - ay') + ay(ay' + bx') - a^2b^2\}^2,$$

which reduces to the required form.

43. Taking (h, k) as the co-ordinates of P, the equation to $A'P$ is

$$y = \frac{k}{h+a}(x+a);$$

and the equation to BD is

$$y - b = \frac{\frac{bh}{a} - b}{-\frac{ak}{b}}x, \quad \text{or } y - b = \frac{b^2(a-h)}{a^2k}x.$$

But since $a^2k^2 = a^2b^2 - b^2h^2$, $\therefore \dfrac{k}{h+a} = \dfrac{b^2(a-h)}{a^2k}$.

Hence BD is parallel to $A'P$.

Similarly AD is parallel to BP.

44. The area of the parallelogram $= BD \times$ (perpendicular distance between BD and OP) $= BD \times$ (difference of perpendiculars from C on BD and OP).

But $$BD^2 = \frac{a^2k^2}{b^2} + \left(b - \frac{bh}{a}\right)^2 = \frac{a^4k^2 + b^4(a-h)^2}{a^2b^2}.$$

Hence the area of the parallelogram is

$$\frac{\sqrt{\{a^4k^2+b^4(a-h)^2\}}}{ab} \times \left\{\frac{a^2kb}{\sqrt{\{a^4k^2+b^4(a-h)^2\}}} - \frac{ab^2(a-h)}{\sqrt{\{a^4k^2+b^4(a-h)^2\}}}\right\}$$

$$=ak+bh-ab.$$

If this result is expressed in terms of the excentric angle, it becomes $ab(\cos\phi+\sin\phi-1)$ or $ab\sqrt{(1+\sin 2\phi)}-ab$; hence the maximum value is $ab(\sqrt{2}-1)$.

45. In the figure to Art. 175, let HQ be drawn to meet PT in Q, so that the angle $HQP=\alpha$.

Now $HZ=HQ\,.\sin\alpha$, and $\angle QHT=ZHT+90^0-\alpha$.

Hence if (r, θ) be the polar co-ordinates of Q, and (r', θ') those of Z we have

$$r'=r\sin a \text{ and } \theta=\theta'+90-\alpha.$$

But, the locus of Z is the circle

$$r'^2+2aer'\cos\theta'=a^2-a^2e^2.$$

Hence the locus of Q is

$$r^2\sin^2\alpha+2aer\sin\alpha\,.\cos(\theta+\alpha-90^0)=a^2-a^2e^2,$$

which is evidently a circle.

Also if $\beta=HPZ$, we have $\sin^2\beta=\frac{HZ^2}{HP^2}=\frac{b^2}{2ar-r^2}=\frac{b^2}{a^2-(a-r)^2}$;

$\therefore$ the minimum value of $\sin^2\beta$ is $\frac{b^2}{a^2}$ or $1-e^2$;

$\therefore$ the maximum value of $\cos\beta$ is e.

Hence if $\cos\alpha$ is less than e, there will be some point at which $\cos\beta=\cos\alpha$, or $\beta=\alpha$; there the lines HP and HQ will coincide, or the circle and ellipse will touch.

If $\cos\alpha$ be greater than e, then β can never $=\alpha$, and so the circle will fall clear of the ellipse externally.

46. Let the conjugate diameters be $y=mx$, and $y=m'x$, so that

$$mm'=-\frac{b^2}{a^2}.$$

The equation to HQ is $\quad y=-\frac{1}{m}(x-ae)$,

and the equation to SQ is $\quad y=-\frac{1}{m'}(x+ae)$.

Multiply these two equations together, and substitute for mm' and we have

$$y^2=-\frac{a^2}{b^2}(x^2-a^2e^2),$$

or

$$b^2y^2+a^2x^2=a^4e^2;$$

which is an ellipse concentric with the given one, and of the same excentricity.

47. Let the ellipses have semi-axes (a, b) and (c, d); then, since the foci are coincident, we have

$$a^2 - b^2 = c^2 - d^2, \text{ or } b^2 + c^2 = a^2 + d^2.$$

Let the equation to one tangent be $y = mx + \sqrt{(a^2m^2 + b^2)}$,

or
$$y - mx = \sqrt{(a^2m^2 + b^2)}.$$

The other tangent will be $my + x = \sqrt{(c^2 + m^2d^2)}$.

Square the two equations and add; we get

$$\begin{aligned}(m^2+1)(x^2+y^2) &= a^2m^2 + d^2m^2 + b^2 + c^2 \\ &= a^2m^2 + d^2m^2 + a^2 + d^2,\end{aligned}$$

or
$$x^2 + y^2 = a^2 + d^2;$$

a circle concentric with the ellipses.

48. Referring to Art. 198, we see that the equation must represent an ellipse referred to *equal* conjugate diameters as axes; also by Art. 193, it is seen that the square of each of the equal conjugate diameters is half the sum of the squares of the semi-axes, or $c^2 = \frac{1}{2}(a^2 + b^2)$, if a, b are the semi-axes.

49. Let the line $y = mx$ meet the ellipse at (h, k);

$$\therefore\ k = mh.$$

The tangent at (h, k) is $a'^2yk + b'^2xh = a'^2b'^2$; (Art. 200).

If this is parallel to $y = m'x$, we have

$$m' = -\frac{b'^2}{a'^2} \cdot \frac{h}{k} = -\frac{b'^2}{a'^2} \cdot \frac{1}{m};$$

$$\therefore\ mm' = -\frac{b'^2}{a'^2}.$$

50. By example 48, the equation to the ellipse may be written

$$x^2 + y^2 = \tfrac{1}{2}(a^2 + b^2).$$

Let (h, k) be any point, then the tangent at this point is

$$xh + yk = \tfrac{1}{2}(a^2 + b^2) \quad \ldots\ldots\ldots\ldots \text{(I)}.$$

Also let the normal be

$$y - k = m(x - h) \quad \ldots\ldots\ldots\ldots \text{(II)}.$$

Since (I) and (II) are perpendicular, we have by Art. 56,

$$1 + \left(m - \frac{h}{k}\right)\cos\alpha - \frac{mh}{k} = 0;$$

if α is the angle between the equal conjugate diameters;

$$\therefore\ m = \frac{k - h\cos\alpha}{h - k\cos\alpha}.$$

But, by Art. 195, $\sin\alpha=\dfrac{2ab}{a^2+b^2}$, $\therefore\ \cos\alpha=\dfrac{a^2-b^2}{a^2+b^2}$.

Hence we find $m=\dfrac{k(a^2+b^2)-h(a^2-b^2)}{h(a^2+b^2)-k(a^2-b^2)}$, and inserting this value in equation (II) we have the required equation.

51. Let the equal conjugate diameters be axes, and let α be the angle between them. Let co-ordinates of P be (h, k). Then the equation to PN is by Art. 56,

$$y-k=-\frac{1}{\cos\alpha}(x-h).$$

Put $y=0$, and we get $\quad CN=h+k\cos\alpha$.

Similarly $\quad CM=k+h\cos\alpha$.

Hence the co-ordinates of the middle point of MN are

$$\frac{h+k\cos\alpha}{2},\ \frac{k+h\cos\alpha}{2}.$$

Now the normal at P is by the last example $y-k=\dfrac{k-h\cos\alpha}{h-k\cos\alpha}(x-h)$, and this is satisfied by the aforesaid co-ordinates.

52. Let diameters CD, CE, CF be parallel to QR, PR, PQ.

Now, if $Pr\,.\,Pr'=k\,.\,CF^2$, then by Art. 208, $Pq\,.\,Pq'=k\,.\,CE^2$. Similarly we may take $Qr\,.\,Qr'=l\,.\,CF^2$, and $Qp\,.\,Qp'=l\,.\,CD^2$;

and $$Rp\,.\,Rp'=mCD^2,\ Rq\,.\,Rq'=mCE^2.$$

Hence the result is obviously true.

The same mode of proof will evidently apply to a polygon.

Also if r coincides with r', &c., the result becomes

$$Pr\,.\,Qp\,.\,Rq\,\ldots\ldots=Pq\,.\,Qr\,.\,Rp\,\ldots\ldots$$

or the product of one set of alternate segments is equal to the product of the other set.

CHAPTER XI.

1. Referring to the figure to Art. 209, we have

$$CH=2CA,\ \text{ or }\ ea=2a;$$
$$\therefore\ e=2;$$

therefore $b^2=3a^2$, and consequently the equation required is

$$a^2y^2-3a^2x^2=-3a^4,\ \text{ or }\ y^2-3x^2=-3a^2.$$

2. Let the co-ordinates of Q be (x, y).

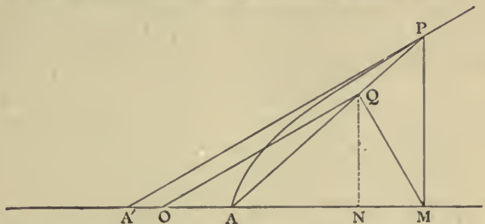
Now $$QM=SP=e\,.\,CM-CA,$$

or $y=ex-a$, a straight line.

3. Since we have to prove that

$$AO : OA' :: AC^2 : BC^2$$
$$:: AQ : QP,$$

it is evident that we have to prove that $A'P$ is parallel to OQ.



Let the co-ordinates of P be (x, y): hence, by hypothesis, we have

$$QN = \frac{a^2y}{a^2+b^2}, \text{ and } AN = \frac{a^2 . AM}{a^2+b^2};$$

hence
$$NM = \frac{b^2(x-a)}{a^2+b^2}.$$

Also, by Euclid VI. 8,

$$\frac{ON}{NQ} = \frac{NQ}{NM} = \frac{a^2y}{b^2(x-a)} = \frac{x+a}{y},$$

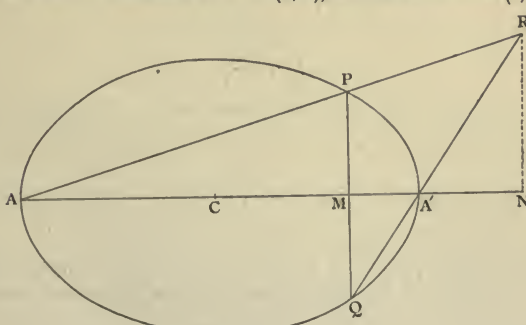
since by equation to the curve

$$b^2(x^2-a^2) = a^2y^2;$$

$$\therefore \frac{ON}{NQ} = \frac{A'M}{PM};$$

and therefore the triangles ONQ, $A'MP$ are similar; whence it is evident that OQ is parallel to $A'P$.

4. Let the co-ordinates of P be (h, k), and those of R be (x, y); and



a, b be the semi-axes of the ellipse, so that

$$a^2k^2 + b^2h^2 = a^2b^2.$$

Now $$\frac{a+x}{y} = \frac{AN}{NR} = \frac{AM}{MP} = \frac{a+h}{k}.$$

Similarly $$\frac{x-a}{y} = \frac{A'N}{NR} = \frac{A'M}{MQ} = \frac{a-h}{k};$$

multiply the two results together, and we get

$$\frac{x^2-a^2}{y^2} = \frac{a^2-h^2}{k^2} = \frac{a^2}{b^2};$$

$$\therefore\quad a^2y^2 - b^2x^2 = -a^2b^2,$$

which is an hyperbola with the same axes as the ellipse.

5. Let the co-ordinates of P be (h, k), and of Q be (x', y').

The equation $k(x^2+y^2-a^2) = y(h^2+k^2-a^2)$ evidently represents the circle named in the question.

The equation to MP is $x = h$; combining this with the equation to the circle, we get $$x' = h,\ y' = \frac{h^2-a^2}{k};$$

therefore by equation to the hyperbola

$$y' = \frac{a^2k}{b^2}.$$

Hence $\dfrac{x'^2}{a^2} - \dfrac{b^2y'^2}{a^4} = 1$, which is an hyperbola whose conjugate axis is $\dfrac{a^2}{b}$.

6. Let the co-ordinates of the point be (h, k); then the equation to the chord of contact is

$$a^2yk + b^2xh - a^2b^2 = 0.$$

The lengths of the two perpendiculars from the foci on this chord are

$$\frac{a^2b^2+b^2hae}{\sqrt{(a^4k^2+b^4h^2)}},\ \text{and}\ \frac{a^2b^2-b^2hae}{\sqrt{(a^4k^2+b^4h^2)}}.$$

If the product of these is equal to a constant m, we get

$$a^4k^2 + b^4h^2\left(1+\frac{a^2e^2}{m}\right) = \frac{a^4b^4}{m}.$$

If m is positive this is an ellipse; if m is negative and less than a^2e^2, it is an hyperbola; if m is negative and greater than a^2e^2, the locus is impossible, as the left-hand side of the equation becomes positive, and the right-hand side negative.

7. Let P be a point of intersection of the two curves, S and H the common foci.

Draw PG bisecting the angle SPH.

Then by Art. 178, PG is the normal to the ellipse at P; and by Art. 228, PG is the tangent to the hyperbola at P. Hence the curves cut at right angles.

8. Let us write the equation in the shape

$$\sqrt{(x^2+y^2)} = k\sqrt{(A^2+B^2)}\left\{\frac{Ax+By+C}{\sqrt{(A^2+B^2)}}\right\}.$$

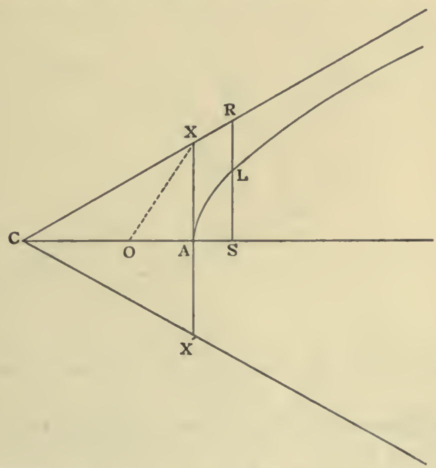
Now the left-hand side is the distance of any point (x, y) from the origin; and the expression $\frac{Ax+By+C}{\sqrt{(A^2+B^2)}}$ is the distance of the same point from the straight line $$Ax+By+C=0.$$

Hence the above equation represents the locus of a point whose distance from a certain fixed point (the origin) is in a constant ratio to its distance from a fixed straight line; it is therefore a conic section, and will be an ellipse or hyperbola according as the fixed ratio

$$k\sqrt{(A^2+B^2)} \text{ is } \lesssim 1.$$

CHAPTER XII.

1. Bisect the angle CXA by OX; then OA is the radius of the circle.



Now $$OA = \frac{\text{area of } CXX'}{\text{semi-perimeter of do.}} = \frac{CA \cdot AX}{CX+AX}.$$

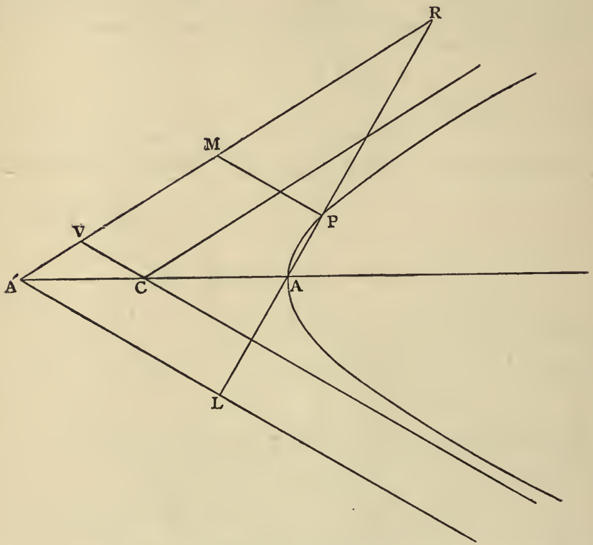
But $$CA=a, \text{ and } AX=b; \therefore CX=ae;$$

$$\therefore OA = \frac{ab}{ae+b} = \frac{ab(ae-b)}{a^2e^2-b^2} = \frac{b(ae-b)}{a}.$$

Also $$SR : SC :: AX : AC;$$
$$\therefore\ SR = be.$$

Again $$SL = \frac{b^2}{a};\ \therefore\ LR = be - \frac{b^2}{a} = \frac{b(ae-b)}{a};$$
$$\therefore\ LR = OA.$$

2. It is easily seen that
$$CV = \tfrac{1}{2}\sqrt{(a^2+b^2)} = c \text{ suppose.}$$



The equation to the hyperbola referred to the asymptotes as axes is, by Art. 261, $$xy = c^2.$$

Let the co-ordinates of P be (h, k); those of A are (c, c).

The equation to PA is
$$y - c = \frac{k-c}{h-c}(x-c);$$
and the equation to $A'R$ is $$x = -c.$$

Hence the ordinate of R, viz. VR, is found to be equal to
$$c + \frac{k-c}{h-c}(-c-c),$$

or $$VR=\frac{hc-c^2-2ck+2c^2}{h-c};$$

$$\therefore \quad MR=\frac{hc-c^2-2ck+2c^2}{h-c}-k=\frac{hc-ck}{h-c}, \text{ since } hk=c^2.$$

Also $$MA'=k+c=\frac{(k+c)(h-c)}{h-c}=\frac{hc-ck}{h-c};$$

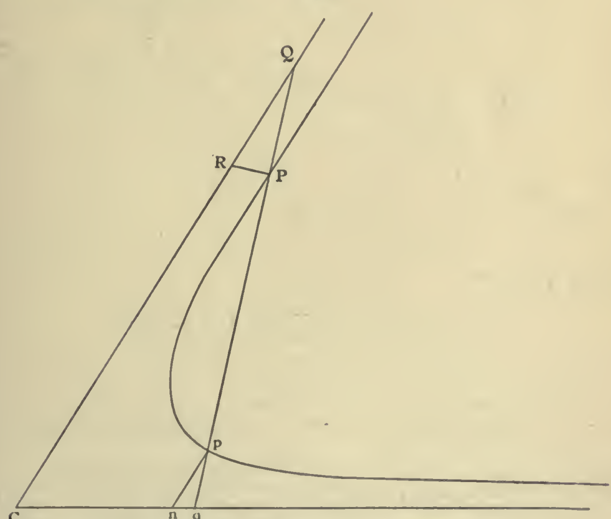
$$\therefore \quad MR=MA'; \quad \therefore \; PR=PL.$$

3. Draw PR parallel to Cq, and pn parallel to CQ.

Take the asymptotes as axes, and let the equation to Pp be

$$y=mx-c.$$

Put $y=0$ in this, and we get $Cq=\frac{c}{m}$.



Let (x_1, y_1) be co-ordinates of P, and (x_2, y_2) of p.

Now solving the equation $xy=\frac{a^2+b^2}{4}$ simultaneously with the equation $y=mx-c$, we get

$$mx^2-cx=\frac{a^2+b^2}{4}.$$

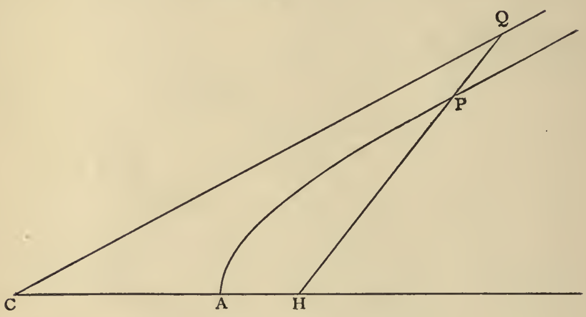
Hence $$x_1+x_2=\frac{c}{m}, \quad \text{or} \quad x_1=\frac{c}{m}-x_2,$$

or $$PR=Cq-Cn=nq.$$

Hence the triangles PQR, pnq are similar and have one side equal; hence they are equal in all respects, or $PQ=pq$.

4. Since $\tan\alpha=\frac{b}{a}$, it is easily seen that

$$\cos\alpha=\frac{1}{e}.$$



Now, by Art. 264,

$$HP=\frac{a(e^2-1)}{1-e\cos\theta}=\frac{ae\left(1-\frac{1}{e^2}\right)}{\frac{1}{e}-\cos\theta}=\frac{a\sec\alpha\,.\,\sin^2\alpha}{\cos\alpha-\cos\theta}$$

$$=\frac{a\sec\alpha\,.\,\sin^2\alpha\,.\,(\cos\alpha+\cos\theta)}{\cos^2\alpha-\cos^2\theta}=\frac{a\sin^2\alpha+a\sec\alpha\,.\,\sin^2\alpha\,.\,\cos\theta}{\sin^2\theta-\sin^2\alpha}.$$

Also $$HQ=CH\,.\,\frac{\sin\alpha}{\sin(\theta-\alpha)}=\frac{ae\sin\alpha}{\sin(\theta-\alpha)}=\frac{a\sec\alpha\,.\,\sin\alpha\,.\,\sin(\theta+\alpha)}{\sin^2\theta-\sin^2\alpha};$$

$$\therefore\quad PQ=HQ-HP=\frac{a\sin\alpha\,.\,\sin\theta-a\sin^2\alpha}{\sin^2\theta-\sin^2\alpha}=\frac{a\sin\alpha}{\sin\theta-\sin\alpha}.$$

5. Take the diameter AB, and the diameter parallel to PQ as axes; let the angle of inclination be α.

Let co-ordinates of P be (h, k_1), of Q be (h, k_2), of A be (a, o), and of B be $(-a, o)$.

The equation to AP is $\qquad y=\frac{k_1}{h-a}(x-a),$

and the equation to BQ is $\qquad y=\frac{k_2}{h+a}(x+a).$

Multiply, and we get $\qquad y^2=\frac{k_1k_2}{h^2-a^2}(x^2-a^2).$

But the equation to the circle is

$$x^2+y^2+2xy\cos\alpha=a^2;$$

and when $x=h$, the corresponding values of y are k_1 and k_2; hence (Introd. § II.), we have

$$k_1k_2=h^2-a^2.$$

Hence our previous equation reduces to

$$y^2=x^2-a^2,$$

which is a rectangular hyperbola.

6. Let e_1 be the excentricity of the conjugate hyperbola, then

$$(e_1^2-1)\,b^2=a^2, \quad \therefore \quad e_1^2=\frac{a^2+b^2}{b^2}=\frac{e^2a^2}{b^2};$$

$$\therefore \quad e_1=\frac{ea}{b}.$$

Now $$SP\sim S'P'=(ex-a)\sim\left(e_1\,.\,\frac{bx}{a}-b\right)$$
$$=(ex-a)\sim(ex-b)$$
$$=a\sim b.$$

7. The values $x=a\sec\theta$, $y=b\tan\theta$, if substituted in the equation

$$a^2y^2-b^2x^2=-a^2b^2,$$

will satisfy this equation.

Also the quantities $a\sec\theta$ and $b\tan\theta$ are capable of expressing magnitudes however large, and have as their numerically minimum values the quantities a and o respectively. Hence they are suitable expressions for the abscissa and ordinate of any point.

8. The equation to the hyperbola is

$$a^2y^2-b^2x^2=-a^2b^2,$$

and to the conjugate it is $\quad a^2y^2-b^2x^2=a^2b^2.$

Let the co-ordinates of P be (c, k), and of Q be (c, l).

The normal at P is $\qquad y-k=-\frac{a^2k}{b^2c}(x-c)$,

and normal at Q is $\qquad y-l=-\frac{a^2l}{b^2c}(x-c)$;

solving simultaneously, we get $y=0$, or the point of intersection is on the axis of x.

Also the tangent at P is $a^2yk-b^2xc=-a^2b^2$,

and the tangent at Q is $\qquad a^2yl-b^2xc=a^2b^2$.

Also since P and Q are on the curves, we have

$$a^2k^2-b^2c^2=-a^2b^2, \quad \text{and} \quad a^2l^2-b^2c^2=a^2b^2.$$

From these four equations we have to eliminate k, l, c.

From the first and second equations, express k and l in terms of c, and substitute in the third and fourth equations, we then get

$$b^2(xc-a^2)^2-a^2c^2y^2=-a^4y^2,$$

and
$$b^2(xc+a^2)^2-a^2c^2y^2=a^4y^2.$$

Adding these we get

$$2b^2x^2c^2+2a^4b^2-2a^2c^2y^2=0.$$

Subtracting them, we get $\quad 4b^2a^2xc=2a^4y^2$.

Eliminating c between these two, we get

$$a^2y^6-b^2x^2y^4=4b^6x^2.$$

9. Let (h, k) be the point from which the tangents are drawn; the equation to the chord of contact is

$$a^2ky-b^2xh=-a^2b^2,$$

which is obviously the tangent to the conjugate hyperbola

$$a^2y^2-b^2x^2=a^2b^2$$

at the point $(-h, -k)$.

10. The equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\left(\frac{xh}{a^2}-\frac{yk}{b^2}\right)^2$ represents some locus passing through the intersection of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, with the chord of contact $\frac{xh}{a^2}-\frac{yk}{b^2}=1$; also it is evidently of the form of two straight lines through the origin; hence it is the required equation.

Also, the general form of the equation to two straight lines through the origin perpendicular to one another is

$$(y-mx)\left(y+\frac{x}{m}\right)=0,$$

or
$$y^2+Axy-x^2=0.$$

Comparing this with the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\left(\frac{xh}{a^2}-\frac{yk}{b^2}\right)^2$ we get the condition

$$\frac{1}{a^2}-\frac{h^2}{a^4}-\frac{1}{b^2}-\frac{k^2}{b^4}=0.$$

Also (h, k) is on the conjugate, so that $\frac{h^2}{a^2}-\frac{k^2}{b^2}=-1$; from these two equations we can find h, k.

11. The equation to a parabola referred to the directrix as axis of y is

$$y^2=2p\left(x-\frac{p}{2}\right),$$

where p is the distance of the focus from the directrix.

The equation to a tangent is $y=m\left(x-\frac{p}{2}\right)+\frac{p}{2m}$; and if this passes through the point (h, k), we have

$$k=m\left(h-\frac{p}{2}\right)+\frac{p}{2m}.$$

This equation if solved as a quadratic in m will give two values m_1 and m_2, belonging to the two tangents which can be drawn through (h, k).

Hence, Introd. § II., we have

$$m_1+m_2=\frac{2k}{2h-p},$$

and

$$m_1m_2=\frac{p}{2h-p}.$$

But

$$\tan^2\alpha=\left(\frac{m_1-m_2}{1+m_1m_2}\right)^2 \text{ (by Art. 41)}$$

$$=\frac{k^2-2hp+p^2}{h^2};$$

$$\therefore\ k^2+(h-p)^2=h^2\sec^2\alpha,$$

or (h, k) is on the hyperbola $y^2+(x-p)^2=x^2\sec^2\alpha$, which is an hyperbola with the same focus and directrix. See Art. 209.

12. It is obvious that both diameters are to *meet* the curve; this evidently becomes more and more possible as the ratio of the conjugate to the transverse axis increases. If therefore we find the value of $\frac{b}{a}$ such that two diameters at right angles approach the limiting position of tangency,—or in other words coincide with the asymptotes,—then any greater value of $\frac{b}{a}$ will make the proposition possible.

But if the asymptotes are to be two lines perpendicular to each other, it follows that $\frac{b}{a}=\tan 45^0=1$; hence if $b>a$ the proposition is possible.

To demonstrate the truth of the proposition in this case.

Let the extremities of the diameters be P and Q, their co-ordinates being (h, k) and (x, y).

Let the perpendicular on PQ be CR.

Then $$CR^2=\frac{CP^2 \cdot CQ^2}{PQ^2}=\frac{(x^2+y^2)(h^2+k^2)}{x^2+y^2+h^2+k^2}=\frac{x^2+y^2}{1+\frac{x^2+y^2}{h^2+k^2}}.$$

But since CP and CQ are at right angles,

$$\frac{x}{y}=-\frac{k}{h};$$

$$\therefore \frac{x^2+y^2}{h^2+k^2}=\frac{x^2}{k^2}=\frac{x^2(a^2k^2-b^2h^2)}{-k^2a^2b^2}$$

$$=\frac{x^2\left(a^2-b^2 \cdot \frac{h^2}{k^2}\right)}{-a^2b^2}=\frac{x^2\left(a^2-b^2 \cdot \frac{y^2}{x^2}\right)}{-a^2b^2}=\frac{b^2y^2-a^2x^2}{a^2b^2};$$

$$\therefore CR^2=\frac{x^2+y^2}{1+\frac{b^2y^2-a^2x^2}{a^2b^2}}=\frac{a^2b^2(x^2+y^2)}{b^2x^2-a^2y^2+b^2y^2-a^2x^2}=\frac{a^2b^2}{b^2-a^2};$$

a constant quantity.

[*Note*. Since CR^2 must be a positive finite quantity, it is evident that b^2 must be $>a^2$, as was otherwise proved above.]

13. Refer the curve to the asymptotes as axes; let CP, CD be the conjugate diameters, and let the co-ordinates of P be (h, k).

Then the equation to CP is $y=\frac{k}{h}x$; and the equation to CD, which is parallel to the tangent at P, is by Art. 262,

$$y=-\frac{k}{h}x.$$

Hence (see Art. 23), the ratio of the sines of the angles made by each of these lines with the axes is the same numerically, namely $\frac{k}{h}$.

14. Let the equation to the ellipse, referred to its conjugate diameters as axes, be

$$\frac{x^2}{c^2}+\frac{y^2}{d^2}=1.$$

And let the hyperbola be $xy=m^2$, and the conjugate hyperbola

$$xy=-m^2.$$

Solving the first and third equations simultaneously, we get

$$\frac{x^2}{c^2}+\frac{m^4}{d^2x^2}=1.$$

If the hyperbola *touches* the ellipse, the two values of x^2 from this equation must be identical; this gives

$$4m^4=c^2d^2.$$

But this is evidently also the condition that the conjugate and ellipse should touch.

Solving the equation $\frac{x^2}{c^2}+\frac{m^4}{x^2d^2}=1$, by the aid of the condition

$$4m^4=c^2d^2,$$

we get
$$x=\pm\frac{c}{\sqrt{2}},\quad y=\pm\frac{d}{\sqrt{2}}.$$

Hence one common diameter is $y=\frac{d}{c}x$.

Similarly the other common diameter is $y=-\frac{d}{c}x$.

And, by the preceding example, we see that these are conjugate to one another.

CHAPTER XIII.

1. By Art. 277, we see that—since the co-ordinates of the centre reduce to $\frac{0}{0}$, that is, are indeterminate—the equation represents two parallel straight lines, namely $x-2y=0$, and $x-2y-2a=0$.

Hence $x-2y-a=0$ is the line of centres.

2. The equations determining the centre are

$$2cx+by-bc=0,\quad\text{and}\quad cx+2by-bc=0,$$

hence the centre is
$$(\tfrac{1}{3}b,\ \tfrac{1}{3}c).$$

3. The equation is $\quad ax^2+2\sqrt{ac}\,xy+cy^2=1,$

or
$$(\sqrt{a}x+\sqrt{c}y+1)(\sqrt{a}x+\sqrt{c}y-1)=0;$$

hence it represents two parallel straight lines.

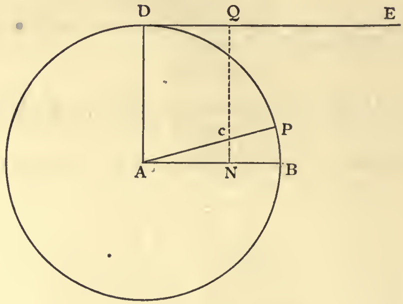
4. Let AB be the fixed radius, and C one position of the moving centre.

Let $AB=a$, and take it as axis of x. Let co-ordinates of C be (x, y).

Now $\quad CP=CN,\ \therefore\ CA=a-CP=a-CN=a-y.$

But $$CA^2 = CN^2 + AN^2;$$

$$\therefore \quad (a-y)^2 = y^2 + x^2,$$



or $$x^2 = a^2 - 2ay,$$

which is a parabola with A as focus, and DE as directrix.

4. (*Aliter.*) $$AP = AD = QN,$$

and $$CP = CN;$$

$$\therefore \quad AC = CQ.$$

Hence by definition of a parabola we get the same result as before.

5. Take AC as axis of x.

Let the co-ordinates of P be $(h, h\tan A)$.

Then equation to CP is $$y = \frac{h\tan A}{h-b}(x-b),$$

and equation to BQ is $$y = \frac{c\sin A}{c\cos A - h}(x-h).$$

Eliminating h, we get

$$c\sin^2 A \,.\, x^2 - 2c\sin A \,.\, \cos A \,.\, xy + (c \,.\, \cos^2 A - b\cos A)\,y^2 + \&c. = 0.$$

Hence the expression corresponding to "$b^2 - 4ac$" in the formulæ of Art. 272, is $$4bc\sin^2 A\cos A.$$

If A is $> \frac{\pi}{2}$, this expression is negative, and the curve is an ellipse.

If A is $< \frac{\pi}{2}$ it is positive, and the curve is an hyperbola.

If $A = \frac{\pi}{2}$, the equation reduces to $x = 0$.

6. Let the co-ordinates of D and E be $(-h, k)$ and (h, k).

The equation to AD is $\quad y = \frac{k}{-h-a}(x-a),$

and equation to CE is $\quad y = \frac{k}{h}x.$

Also $\quad a^2k^2 + b^2h^2 = a^2b^2.$

Eliminating h, k between these equations, we get

$$a^2y^2 - 3b^2x^2 + 4ab^2x - a^2b^2 = 0,$$

which is the equation to an hyperbola.

Solving with respect to y, we get

$$ay = \pm\sqrt{3}\,bx\left\{1 - \frac{4a}{3x} + \frac{a^2}{3x^2}\right\}^{\frac{1}{2}}.$$

Hence, Art. 279, II. the asymptotes are

$$ay = \pm\sqrt{3}\left(bx - \frac{2ab}{3}\right).$$

7. Let the semi-axes of the ellipses be (a, b) and (c, d).

Let co-ordinates of P be (h, k), and let P lie on the line

$$y = mx + n.$$

The chords of contact are

$$a^2yk + b^2xh = a^2b^2,$$

and $\quad c^2yk + d^2xh = c^2d^2.$

Also we have $\quad k = mh + n.$

Eliminating h, k between these equations, we get

$$(a^2d^2 - b^2c^2)\,nxy + (d^2 - b^2)\,a^2c^2my + (c^2 - a^2)\,b^2d^2x = 0,$$

and by Art. 274 this is a rectangular hyperbola.

8. In the previous example, let $a = c$.

The locus becomes $\quad nxy + a^2my = 0.$

Hence the locus is either

$$nx + a^2m = 0, \text{ or else } y = 0.$$

But it is evidently the latter, when P is on the tangent at the extremity of a or c.

Hence the locus reduces to the common axis.

9. Take the fixed point as origin, and let the axes be parallel to the two intersecting straight lines.

Let α be the inclination of the two lines.

Then, by supposition, if (x, y) be the point of intersection,

$$x \sin\alpha \times y \sin\alpha = \text{constant} = c^2 \text{ suppose};$$

$$\therefore\quad xy = c^2 . \operatorname{cosec}^2\alpha, \text{ an hyperbola.}$$

10. Let the straight lines make angles α and $\alpha+2\beta$ with the major axis, so that 2β is the constant included angle.

The polar equations to the two tangents are

$$r=\frac{l}{e\cos\theta+\cos(\alpha-\theta)},\quad\text{and}\quad r=\frac{l}{e\cos\theta+\cos(\alpha+2\beta-\theta)};$$

where these meet we have

$$\cos(\alpha-\theta)=\cos(\alpha+2\beta-\theta),$$

and hence we get $$\theta=\alpha+\beta.$$

Hence the locus is $$r=\frac{l}{e\cos\theta+\cos\beta},$$

or $$r=\frac{l\sec\beta}{1+e\sec\beta\,.\cos\theta},$$

which is an ellipse or hyperbola according as $e\sec\beta\lesssim 1$.

11. Turning the axes through an angle of 45^0, the equation becomes

$$y^2+\frac{a}{2\sqrt{2}}y=\frac{a}{2\sqrt{2}}x.$$

Hence the latus rectum is $\frac{a}{2\sqrt{2}}$ or $\frac{a\sqrt{2}}{4}$.

12. Let the axes be turned round, so that the equation becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

therefore by Art. 274 $$0-\frac{4}{a^2b^2}=4-4\times\tfrac{5}{4};$$

$$\therefore\ ab=2.$$

13. Solving with respect to x, we get

$$x-\frac{y}{2b}=\pm\frac{y}{2b}\left\{1-\frac{4bc}{y^2}\right\}^{\frac{1}{2}}$$

$$=\pm\frac{y}{2b}\left\{1-\frac{2bc}{y^2}+\&c.\dots\right\};$$

therefore the asymptotes are $x=0$ and $x=\frac{y}{b}$.

Hence the tangent of angle between the asymptotes is $\frac{1}{b}$.

14. The general equation to a parabola is

$$ax^2 + 2\sqrt{ac}\,xy + cy^2 + dx + ey + f = 0.$$

Put $y=0$, and we get $\qquad ax^2 + dx + f = 0.$

The roots of this quadratic are to be each equal to α, so that

$$\frac{d}{a} = -2\alpha, \text{ and } \frac{f}{a} = \alpha^2.$$

Put $x=0$, and we get $\qquad cy^2 + ey + f = 0.$

The roots of this are to be β and β', so that

$$-\frac{e}{c} = \beta + \beta', \text{ and } \frac{f}{c} = \beta\beta'.$$

Hence the original equation becomes

$$\beta\beta' x^2 + 2\alpha\sqrt{\beta\beta'}\,xy + \alpha^2 y^2 - 2\alpha\beta\beta' x - \alpha^2(\beta+\beta')\,y + \alpha^2\beta\beta' = 0.$$

15. The general equation to a rectangular hyperbola may be written, by Art. 274,

$$Ax^2 + Bxy - Ay^2 + Dx + Ey + 1 = 0.$$

Let the given points be $(h, 0)$; $(k, 0)$; $(0, l)$; $(0, m)$.

We have therefore

$$\left.\begin{aligned} Ah^2 + Dh + 1 &= 0 \\ Ak^2 + Dk + 1 &= 0 \\ -Al^2 + El + 1 &= 0 \\ -Am^2 + Em + 1 &= 0 \end{aligned}\right\} \dots\dots\dots\dots (\text{I}).$$

These four equations will determine A, D, E, and also the relation that must exist between h, k, l, m; but we have no data to determine B, so that the position of the hyperbola is indefinite.

By Art. 270, if (x, y) be the centre of the hyperbola, we have

$$x = \frac{-2AD - BE}{B^2 + 4A^2}, \text{ and } y = \frac{2AE - BD}{B^2 + 4A^2};$$

$$\therefore\ x^2 + y^2 = \frac{4A^2E^2 + B^2D^2 + 4A^2D^2 + B^2E^2}{(B^2 + 4A^2)^2} = \frac{E^2 + D^2}{B^2 + 4A^2}$$

$$= -\frac{D}{2A}x + \frac{E}{2A}y;$$

$$\therefore\ x^2 + y^2 + \frac{D}{2A}x - \frac{E}{2A}y = 0,$$

which is a circle passing through the origin.

From the first two equations of (I) it is evident that h, k are roots of the same quadratic, so that

$$h + k = -\frac{D}{A}, \text{ and } hk = \frac{1}{A}.$$

Hence the co-ordinates of the point midway between $(h, 0)$ and $(k, 0)$ will be $\left(-\frac{D}{2A}, 0\right)$, which evidently lies on the circle.

Similarly the point midway between $(0, l)$ and $(0, m)$ lies on the circle.

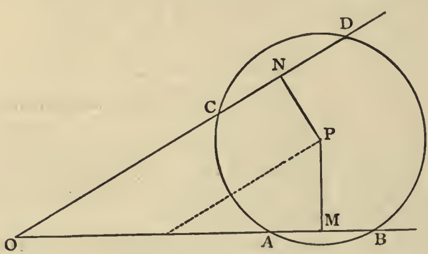
Also in the equation to the circle put $\frac{h}{2}$ for x, and $\frac{l}{2}$ for y, and we get on the left-hand side

$$\frac{Ah^2+Dh}{4A} - \frac{-Al^2+El}{4A},$$

which by (I) is evidently $=0$.

Hence the circle goes through the point midway between $(h, 0)$ and $(0, l)$; similarly for the other point.

16. Take the two fixed axes as axes of co-ordinates, and let them meet at an angle α.



Let $$AB=2a, \quad CD=2b.$$

Take (x, y) as co-ordinates of P;

$$\therefore \; OM=x+y\cos\alpha,$$

$$ON=y+x\cos\alpha.$$

But, by Euclid, $$OA \, . \, OB=OC \, . \, OD,$$

or $$OM^2-AM^2=ON^2-CN^2;$$

$$\therefore \; x^2+y^2\cos^2\alpha+2xy\cos\alpha-a^2=y^2+x^2\cos^2\alpha+2xy\cos\alpha-b^2,$$

or $$x^2-y^2=(a^2-b^2)\operatorname{cosec}^2\alpha;$$

a rectangular hyperbola.

17. Let the common focus be S, and let H be the other focus of the fixed ellipse, and S' of the variable one.

Let P be the point of contact; then the normal at P bisects the angle between SP and HP, and also the angle between SP and $S'P$; hence HP and $S'P$ coincide in direction, or S' is always a point on HP.

(I). When the major axis is given, we have $SP+S'P=$constant; also $SP+HP=$constant; therefore HS' is constant, or H lies on a circle whose centre is H, and radius the difference of the major axes.

(II). If the minor axis is given, then the product of the perpendiculars from S and S' on the tangent at P is constant; but the product of the perpendiculars from S and H is also constant; hence the perpendicular from S' is to the perpendicular from H in a constant ratio; therefore $S'P : HP$ is a constant ratio.

Hence $HS' : HP$ is a constant ratio.

Let us therefore take $HS'=\mu . HP$, and let the polar equation to the fixed ellipse with H as pole be

$$r=\frac{l}{1+e\cos\theta},$$

then the locus of S' is

$$HS'=\frac{\mu l}{1+e\cos\theta},$$

which is an ellipse with H for focus, and of equal excentricity.

18. The given equation may be resolved into

$$(y-3x+1)(y-2x+4)=0,$$

so that it represents two straight lines.

19. Turning the axes through 45^0, we have

$$\frac{1}{2}x^2+\tfrac{3}{2}y^2-\frac{6x}{\sqrt{2}}=0,$$

or

$$(x-3\sqrt{2})^2+3y^2=18,$$

which is an ellipse whose centre is $(3\sqrt{2}, 0)$ and its semi-axes $3\sqrt{2}$ and $\sqrt{6}$.

20. Transferring the origin to (c, c), we have

$$25x^2-8xy+y^2-9c^2=0.$$

Turning the axes through an angle $\frac{1}{2}\tan^{-1}\left(-\frac{1}{3}\right)$, we have

$$(13-4\sqrt{10})\,x^2+(13+4\sqrt{10})\,y^2=9c^2,$$

which is an ellipse whose axes coincide with the axes of co-ordinates.

21. If the axes of co-ordinates be turned through an angle θ, such that $\tan 2\theta = \dfrac{2b}{a-c}$, then the new axes of co-ordinates will (by Art. 271), be the axes of the conic.

Hence with the original axes of co-ordinates, the axes of the conic are

$$y - x\tan\theta = 0 \text{ and } y + x\cot\theta = 0.$$

These can be combined in the one equation

$$y^2 - x^2 + 2xy\cot 2\theta = 0,$$

which reduces to given form.

22. Let the tangent parallel to the given chords be the axis of y, and let its diameter be the axis of x.

Let the equation to the parabola be $y^2 = 4ax$; and let the co-ordinates of the ends of one chord be (h, k) and $(h, -k)$.

The equations to the other sides of the triangle will be of the shape

$$y - k = m(x - h) \text{ and } y + k = n(x - h)$$

where m and n are constants.

Eliminating h and k between these equations by help of the condition $k^2 = 4ah$, we get the equation

$$(n-m)^2y^2 + 8ay(m+n) = 4ax(m+n)^2,$$

which is in general a parabola.

If however the lines $y - k = m(x-h)$ and $y + k = n(x-h)$ are tangents at (h, k) and $(h, -k)$ respectively, we have

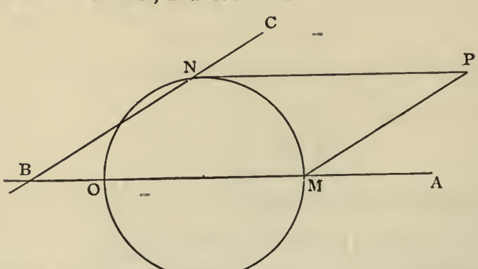
$$m = \frac{2a}{k} \text{ and } n = -\frac{2a}{k};$$

$$\therefore m + n = 0.$$

In this case the locus reduces to $y = 0$.

23. Let the angle between BC and OA be ω.

Take BC and BA as axes, and let $BO = a$



Let the co-ordinates of P be (h, k).

Then (by Art. 104), the equation to the circle is

$$x^2+y^2+2xy\cos\omega-x(a+h)-y(a+h)\cos\omega+ah=0.$$

But when $x=0$, $y=k$, hence we have

$$k^2-k(a+h)\cos\omega+ah=0,$$

which is the equation to the locus of P.

This is evidently an hyperbola unless $\omega=90^0$, in which case the locus is $k^2=-ah$, a parabola.

24. Let the equation to the chord of contact be

$$ky=2a(x+h).$$

Then, by Ex. 56, Chap. VIII., the equation to the lines from the vertex to the ends of this chord is

$$hy^2=2x(ky-2ax).$$

Hence, by Ex. 31, Chap. III.,

$$\tan\beta=\frac{\sqrt{4k^2-16ah}}{4a+h}.$$

Squaring, we get

$$4k^2-h^2\tan^2\beta-8ah(2+\tan^2\beta)-16a^2\tan^2\beta=0,$$

which is an hyperbola in which the ratio of the conjugate axis to the transverse axis is $\frac{1}{2}\tan\beta$;

$$\therefore\ \tan\phi=\frac{1}{2}\tan\beta.$$

25. Let (h, k) be the middle point of the chord whose equation is

$$y=x\tan\theta+p.$$

To find the intersection of this chord with the curve, we have

$$ax^2+2bx(x\tan\theta+p)+c(x\tan\theta+p)^2+2ex+2f(x\tan\theta+p)+g=0.$$

If x_1 and x_2 are the roots of this equation, we have

$$h=\frac{1}{2}(x_1+x_2)=-\frac{bp+cp\tan\theta+e+f\tan\theta}{a+2b\tan\theta+c\tan^2\theta};$$

and $$k=h\tan\theta+p=\frac{ap+bp\tan\theta-e\tan\theta-f\tan^2\theta}{a+2b\tan\theta+c\tan^2\theta}.$$

Eliminating p we get as the locus of (h, k) the equation

$$h(a+b\tan\theta)+k(b+c\tan\theta)+e+f\tan\theta=0\ \ldots\ldots\ldots\ldots\ldots(A).$$

It is easily seen that this represents a straight line through the centre of the given curve as the co-ordinates of the centre are $\frac{ce-bf}{b^2-ac}$, $\frac{af-be}{b^2-ac}$ (see Art. 270).

(I). Let the curve be a parabola; then equation (A) will represent a straight line parallel to the axis of the curve; if therefore we draw a straight line to cut (A) at right angles, and terminated both ways by the curve, then the line bisecting this last one at right angles will be the axis.

(II). If the curve be a central one, then the given line $x\sin\theta - y\cos\theta = 0$ is, by definition, conjugate to the line (A); and we can find the angle between the two conjugate diameters, and then determine what value of θ will make this angle a right angle,—in which case the diameters become the axes.

26. Since the equation may be written

$$(x^2+y^2+xy\sqrt{2}-a^2)(x^2+y^2-xy\sqrt{2}-a^2)=0,$$

it therefore represents the two ellipses

$$x^2+y^2+xy\sqrt{2}=a^2 \text{ and } x^2+y^2-xy\sqrt{2}=a^2.$$

27. Let PB make an angle α with AB, and PC an angle β with AC; take AB and AC as axes; let the co-ordinates of P be represented by x and y. Let ω be the angle CAB. Let $BC=a$.

Now $$PC=\frac{x\sin\omega}{\sin\beta},$$

and $$PB=\frac{y\sin\omega}{\sin\alpha}.$$

Also $$PC^2+PB^2-2PC\,.\,PB\cos CPB=BC^2=a^2;$$

$$\therefore\ x^2\sin^2\alpha+y^2\sin^2\beta-2xy\sin\alpha\,.\,\sin\beta\cos(\alpha+\beta-\omega)=\frac{a^2\sin^2\alpha\,.\,\sin^2\beta}{\sin^2\omega};$$

and, by Art. 272, this is an ellipse.

28. Take the diameter and tangent at the common point as axes; the parabolas may therefore be denoted by

$$y^2=4ax,\quad y^2=4bx,\ \&c.$$

The equations to the parallel tangents will be of the form

$$y=mx+\frac{a}{m},\qquad y=mx+\frac{b}{m},\ \&c.$$

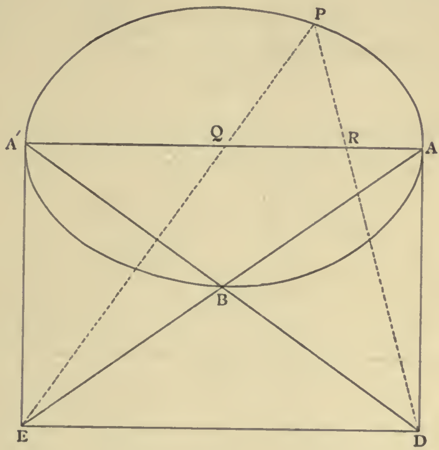
The points of contact of these tangents will be

$$\left(\frac{a}{m^2},\ \frac{2a}{m}\right),\quad \left(\frac{b}{m^2},\ \frac{2b}{m}\right),\ \&c.,$$

and these all lie on the straight line $y=2mx$ passing through the common point.

29. Take A' as origin: the equation to the ellipse will be

$$y^2=\frac{b^2}{a^2}(2ax-x^2).$$



Let the co-ordinates of P be (h, k).

The co-ordinates of E are $(0, -2b)$.

Hence the equation to PE is

$$y+2b=\frac{k+2b}{h}\,.\,x.$$

Put $y=0$, and we get $$A'Q=\frac{2bh}{k+2b}.$$

Similarly we get $$A'R=\frac{2ak+2bh}{k+2b};$$

$$\therefore\ QR=\frac{2ak}{k+2b},\quad\text{and}\quad RA=\frac{4ab-2bh}{k+2b};$$

$$\therefore\ A'Q\,.\,RA=\frac{8ab^2h-4b^2h^2}{(k+2b)^2}=\frac{4a^2k^2}{(k+2b)^2},$$

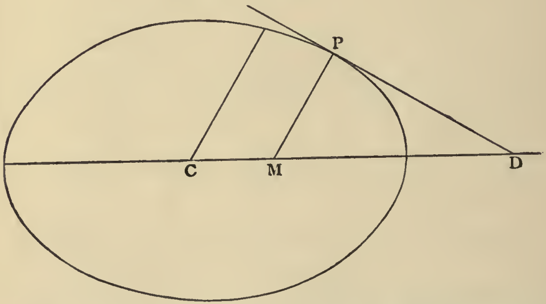
by the equation to the curve.

Hence $$A'Q\,.\,RA=QR^2.$$

30. The equation to an ellipse referred to equal conjugate diameters is of the form
$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1.$$

And the equation to a tangent to it is
$$\frac{xx'}{a^2}+\frac{yy'}{a^2}=1.$$

If this passes through the point $(h, 0)$ we get
$$x'=\frac{a^2}{h},$$
and putting this value in the equation to the curve we get
$$y'=\sqrt{\left(a^2-\frac{a^4}{h^2}\right)}.$$



Hence, in the adjoining figure, if CD be the fixed diameter, and D be the point $(h, 0)$, and DP the tangent, we have proved that the abscissa CM and the ordinate MP are both fixed in length. Hence, whatever the *direction* in which the other conjugate diameter lies, or in other words whatever the direction in which MP is drawn, the locus of P is a circle with M for centre and radius
$$\sqrt{\left(a^2-\frac{a^4}{h^2}\right)}.$$

31. Let CB, CD be the semi-diameters parallel to TP, TQ. Then, by Art. 208,
$$\frac{TP^2}{TQ^2}=\frac{CB^2}{CD^2}=\frac{SP.HP}{SQ.HQ} \text{ by Art. 193.}$$

Also
$$\frac{RP}{RQ}=\frac{SP}{SQ} \text{ and } \frac{R'P}{R'Q}=\frac{HP}{HQ},$$
by the definition of a conic; therefore
$$\frac{TP^2}{TQ^2}=\frac{RP.R'P}{RQ.R'Q}.$$

32. This is fully solved in the Answers.

CHAPTER XIV.

1. Let the two fixed straight lines be OB and OC, and take them as axes. Let the fixed point P through which the variable lines are drawn have $(a, b,)$ as its co-ordinates.

The equation to any straight line through P is of the form

$$y - b = m(x - a).$$

The intercepts of this line on the axes are

$$a - \frac{b}{m} \text{ and } b - am.$$

Hence, if (h, k) be the middle point of this line, we have

$$h = \frac{a}{2} - \frac{b}{2m}, \quad k = \frac{b}{2} - \frac{am}{2}.$$

Eliminating m, we get

$$(2h - a)(2k - b) = ab.$$

Hence the locus is $\qquad (2x - a)(2y - b) = ab.$

If we move the origin to the point midway between O and P, this becomes

$$xy = \tfrac{1}{4}ab,$$

which is an hyperbola referred to its asymptotes.

2. Let the co-ordinates of P be (h, k), and those of Q be (x, y).

Then $\qquad x = h - (a - eh), \quad \text{or} \quad h = \dfrac{x+a}{1+e};$

and $\qquad y = k.$

But $\qquad a^2k^2 + b^2h^2 = a^2b^2;$

$$\therefore \quad a^2y^2 + b^2\left(\frac{x+a}{1+e}\right)^2 = a^2b^2.$$

Move the origin to the left-hand end of major axis, and this becomes

$$a^2y^2 + \frac{b^2x^2}{(1+e)^2} = a^2b^2,$$

which is an ellipse with semi-axes $a(1+e)$ and b.

Similarly for Q'.

3. Let Qp and Pq intersect in R.

Let $AP = a$, and $AQ = b$, and take these as axes.

Let $\qquad Ap : pP :: m : 1;$

$$\therefore \quad Ap = \frac{ma}{m+1}.$$

Similarly $$Aq = \frac{b}{m+1}.$$

Now, the equation to pQ is

$$y - b = -\frac{b(m+1)}{ma}x,$$

and the equation to Pq is

$$y - \frac{b}{m+1} = -\frac{b}{a(m+1)}x.$$

Eliminating m between these two equations, we get

$$(ab - ay - bx)^2 = abxy,$$

or $$a^2y^2 + abxy + b^2x^2 - 2ab^2x - 2a^2by + a^2b^2 = 0;$$

and by Art. 280 this is an ellipse.

Also, by putting $y=0$, we get an equation which gives two *coincident* values of x, namely a; hence the curve touches AP at P. Similarly it touches AQ at Q.

4. Take TP, TQ as axes, and let their lengths be h and k.

Then the equation to the parabola (by Art. 294) is

$$\sqrt{\frac{x}{h}} + \sqrt{\frac{y}{k}} = 1.$$

Also, by Art. 295, the equation to the tangent at (x', y') is

$$\frac{y}{\sqrt{(ky')}} + \frac{x}{\sqrt{(hx')}} = 1.$$

Putting $x=0$, we get $$Tq = \sqrt{(ky')}.$$

Similarly, putting $y=0$, we get

$$Tp = \sqrt{(hx')};$$

$$\therefore \frac{Tp}{TP} + \frac{Tq}{TQ} = \frac{\sqrt{(hx')}}{h} + \frac{\sqrt{(ky')}}{k}$$

$$= \sqrt{\frac{x'}{h}} + \sqrt{\frac{y'}{k}} = 1,$$

since the point (x', y') is on the curve.

5. From the previous example we have

$$\frac{Tp}{TP} = 1 - \frac{Tq}{TQ} = \frac{Qq}{TQ}.$$

But $$TP = TQ;$$

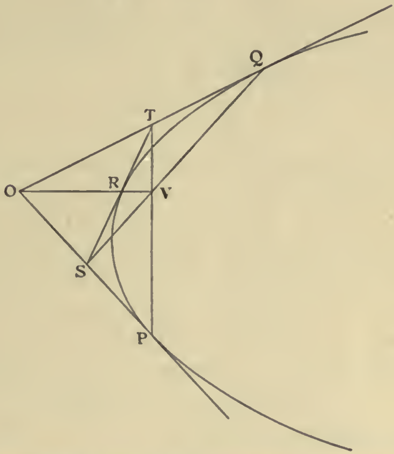
$$\therefore Tp = Qq.$$

Similarly $$Pp = Tq.$$

6. Let $OP=h$, $OQ=k$, and take these as axes.

The equation to the curve is

$$\sqrt{\frac{x}{h}}+\sqrt{\frac{y}{k}}=1.$$



If the co-ordinates of R are (x', y'), the equation to the tangent there is

$$\frac{y}{\sqrt{(ky')}}+\frac{x}{\sqrt{(hx')}}=1.$$

Hence the co-ordinates of T are

$$\{0, \quad \sqrt{(ky')}\}.$$

Similarly the co-ordinates of S are

$$\{\sqrt{(hx')}, \quad 0\}.$$

Now, the equation to PT is

$$hy-h\sqrt{(ky')}+x\sqrt{(ky')}=0,$$

and the equation to SQ is

$$y\sqrt{(hx')}-k\sqrt{(hx')}+kx=0.$$

The co-ordinates of the point of intersection of these two lines are found to be, (remembering that x', y' satisfy the equation to the curve)

$$\frac{x'\sqrt{(hk)}}{\sqrt{(hk)}-\sqrt{(x'y')}} \text{ and } \frac{y'\sqrt{(hk)}}{\sqrt{(hk)}-\sqrt{(x'y')}};$$

and these evidently satisfy the equation to OR, which is

$$y=\frac{y'}{x'}\cdot x.$$

7. Let the given ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and let the external point be (h, k).

Now the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{xh}{a^2}+\frac{yk}{b^2}$ is an ellipse, by Art. 280.

Also it is satisfied at the intersections of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ with } \frac{xh}{a^2}+\frac{yk}{b^2}=1;$$

and it is satisfied by the co-ordinates of (h, k) and $(0, 0)$. Hence it goes through all the specified points.

Also, by Art. 299 (8), it is similar and similarly situated to the given ellipse.

8. Take the common centre as origin, and let one ellipse be

$$a^2y^2+b^2x^2=a^2b^2 \quad \text{.................................(A)},$$

and the other one $$m^2a^2y^2+m^2b^2x^2=m^4a^2b^2 \quad \text{..........................(B)}.$$

Let (h, k) be the centre of the third ellipse, then the equation to this ellipse is

$$p^2a^2(y-k)^2+p^2b^2(x-h)^2=p^4a^2b^2 \quad \text{....................(C)}.$$

The chord of intersection of A and C is

$$a^2\{(y-k)^2-y^2\}+b^2\{(x-h)^2-x^2\}=p^2a^2b^2-a^2b^2,$$

which reduces to $$y=-\frac{b^2h}{a^2k}x+\&c.$$

The tangent to B at the point (h, k) is

$$m^2a^2yk+m^2b^2xh=m^4a^2b^2,$$

or $$y=-\frac{b^2h}{a^2k}x+\&c.$$

Hence the lines are parallel.

9. Since in any conic the tangents at the ends of a focal chord intersect on the corresponding directrix, therefore the directrix is the polar of the focus.

By Art. 287, the equation to any conic with the focus as origin and initial line as an axis is $$r=l-er\cos\theta,$$

which transforms into $$x^2+y^2=(l-ex)^2.$$

This may be written

$$x^2(1-e^2)+y^2+2elx-l^2=0.$$

The polar of any point (x', y') is, by Art. 289,

$$2x\{(1-e^2)x'+el\}+2yy'+2elx'-2l^2=0 \text{..................(I)}.$$

Hence, since the directrix is the polar of (0, 0), its equation is

$$2elx-2l^2=0, \text{ or } ex=l.$$

The intersection of this directrix with equation (I) is easily found to be the point

$$\left(\frac{l}{e}, \quad -\frac{lx'}{ey'}\right).$$

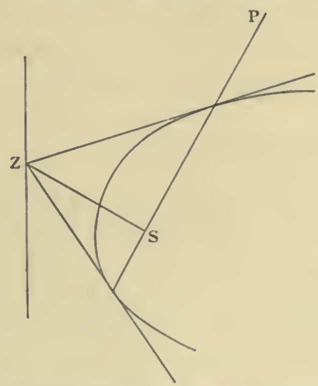
Hence the equation to the line joining this intersection to the focus is

$$y=-\frac{x'}{y'}\cdot x,$$

which is evidently perpendicular to the line

$$y=\frac{y'}{x'}\cdot x.$$

9. (*Aliter*). Since the polar of any point is (by Art. 120) the locus of intersection of tangents at the ends of every chord through the point, therefore the tangents at the ends of the focal chord PS will intersect on the



polar of P; but they intersect (Arts. 156, 207) on the directrix; hence Z their point of intersection is where the polar and directrix intersect. Consequently it is evident (Arts. 156, 207) that PZ subtends a right angle at the focus.

10. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

The equation to a normal is

$$y-y'=\frac{a^2y'}{b^2x'}(x-x').$$

If this goes through a point (h, k), we have

$$k - y' = \frac{a^2 y'}{b^2 x'} (h - x'),$$

or

$$a^2 e^2 x'y' + b^2 x'k - a^2 y'h = 0.$$

Hence the point (x', y') is on the curve

$$a^2 e^2 xy + b^2 xk - a^2 yh = 0,$$

which, by Art. 274, is a rectangular hyperbola, and it is evidently satisfied by (h, k).

Also the terms containing the first powers of x and y can be removed by moving the axes parallel to themselves, and the equation then is of the form given in Art. 261; hence the asymptotes are parallel to the original axes.

11. Take the centre of the circle as origin, and let the co-ordinates of P be (h, k), and those of Q be (x, y).

Let α be the constant angle; therefore

$$x = h + k \cos \alpha, \quad y = k \sin \alpha.$$

Let c be radius of circle; therefore

$$h^2 + k^2 = c^2, \quad \therefore \ (y \operatorname{cosec} \alpha)^2 + (x - y \cot \alpha)^2 = c^2,$$

or

$$x^2 + y^2 (\cot^2 \alpha + \operatorname{cosec}^2 \alpha) - 2xy \cot \alpha = c^2,$$

which is an ellipse.

12. This is fully solved in the Answers.

13. Let the fixed lines meet at O, and let them be axes. Let one particular value of OP be p, and of OQ be q; then the general values of OP and OQ will be mp and mq, where m is any variable quantity whatever.

Let the co-ordinates of H and R be (a, b) and (c, d).

The equation to PH is $\quad y = \dfrac{b}{a - mp} (x - mp),$

and to QR is $\quad y - mq = \dfrac{d - mq}{c} \cdot x.$

Eliminating m between these two equations, we get the required locus, namely, $\quad bqx^2 - cpy^2 + (dp - aq) xy - x (bcq + bdp) + y (acq + bcp) = 0;$

and this evidently goes through $(0, 0)$, and (a, b) and (c, d).

14. Take A as origin, and AB as axis of x. Let AP and BT intersect in Q. Take (h, k) as the co-ordinates of P.

The equation to the circle is

$$(x - a)^2 + y^2 = a^2,$$

and equation to PT is $\quad (x - a)(h - a) + yk = a^2.$

Putting $x=0$, we get $$AT=\frac{ah}{k}.$$

Hence equation of BT is $y=-\frac{h}{2k}(x-2a)$,

and equation to AP is $$y=\frac{k}{h}x.$$

Multiplying the one equation by the other, we get

$$2y^2+x^2-2ax=0,$$

which is an ellipse whose major axis is AB, and excentricity $\frac{1}{\sqrt{2}}$.

15. If the equation $\frac{1}{r}-c\,.\cos\theta=b\,.\cos\left(\theta-\frac{\alpha+\beta}{2}\right)\sec\frac{\alpha-\beta}{2}$ be transformed to Cartesian co-ordinates, it is seen to be a straight line.

Also, if $\theta=\alpha$ the equation becomes $\frac{1}{r}-c\,.\cos\theta=b$, or in other words it is satisfied by the point on the curve for which $\theta=\alpha$.

Similarly for the point at which $\theta=\beta$.

16. Using the notation of the previous example, let 2ω be the constant angle, so that $$2\omega=\alpha-\beta.$$

Hence the equation to the chord, as obtained in the previous example, may be written

$$\frac{1}{r}-c\,.\cos\theta=b\cos\theta\,.\cos(\omega+\beta)\,.\sec\omega+b\sin\theta\,.\sin(\omega+\beta)\sec\omega,$$

or in Cartesian co-ordinates

$$1=cx+bx\,.\cos(\omega+\beta)\sec\omega+by\,.\sin(\omega+\beta)\sec\omega.$$

By Art. 44 the equation to the perpendicular on this from focus is

$$0=cy+by\,.\cos(\omega+\beta)\sec\omega-bx\,.\sin(\omega+\beta)\sec\omega.$$

These equations may be written

$$\sin\beta(by-bx\tan\omega)+\cos\beta(bx+by\tan\omega)=1-cx,$$

and $$\cos\beta(by-bx\tan\omega)-\sin\beta(bx+by\tan\omega)=-cy.$$

Square and add, and we get

$$(by-bx\tan\omega)^2+(bx+by\tan\omega)^2=(1-cx)^2+c^2y^2,$$

or $$(x^2+y^2)(b^2\sec^2\omega-c^2)+2cx=1;$$

which is a circle except when $b\sec\omega=\pm c$, in which case it becomes a straight line.

17. Using the notation of the two preceding examples, the equation to PQ is

$$\frac{1}{r}=c \, . \, \cos\theta+b \, . \sec\omega \, . \cos(\omega+\beta-\theta) \, ;$$

and by Art. 288 this is the tangent to a conic, with the pole as focus, the point of contact being determined by the angular co-ordinate $\omega+\beta$.

18. Let P and Q be the fixed points, and PM, QN the perpendiculars on the directrix. Let S be the focus whose locus is required.

Then, by the definition of a conic,

$$PS : QS :: PM : QN ;$$

therefore $PS : QS$ is a fixed ratio.

Hence, by Ex. 6 of Chap. VI., the locus of S is a circle.

19. Let SO be the perpendicular from the focus on the directrix, and p its length.

Then the equation to the ellipse, with focus as origin, is

$$y^2+x^2=e^2(x+p)^2 \dots\dots\dots\dots\dots\dots\text{(A)}.$$

Let α be the angle made by the chord with the directrix, so that

$$\sin\alpha=e \dots\dots\dots\dots\dots\dots\text{(B)}.$$

Now, the equation to this focal chord is

$$y=x\cot\alpha \dots\dots\dots\dots\dots\dots\text{(C)}.$$

If between these three equations we eliminate α and e, we shall have the required locus.

From (B) and (C) we get $\quad y^2+x^2=\dfrac{x^2}{e^2}$.

Multiply this by (A), and we get

$$(y^2+x^2)^2=x^2(x+p)^2,$$

or

$$(y^2+2x^2+px)(y^2-px)=0.$$

Hence there are two loci, viz. the parabola $y^2-px=0$, whose vextex is S, and the ellipse $y^2+2x^2+px=0$, which has one end of its minor axis at S and the other end at the middle of SO.

20. The equation to a conic referred to focus as pole, and axis as initial line, is

$$r=\frac{l}{1+e\cos\theta}.$$

But if p be the perpendicular from the focus on the directrix, we have by the definition of a conic, $ep=l$.

Hence the equation can be written

$$r=\frac{ep}{1+e\cos\theta}.$$

But, by hypothesis, $r \propto \frac{1}{l}$, or $r = \frac{m}{ep}$, where m is a constant.

Eliminating e between the two equations, we have

$$r^2 + \frac{m}{p} r \cos\theta = m,$$

or
$$x^2 + y^2 + \frac{mx}{p} = m,$$

which is a circle.

21. Let one conic be
$$r = \frac{l}{1 + e\cos\theta} \text{ .... (I)},$$

and the other
$$r = \frac{l'}{1 + e'\cos(\theta - \alpha)} \text{ .... (II)}.$$

The form of the tangent to the first conic at the point (β) is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \beta);$$

and the tangent to the second is

$$\frac{l'}{r} = e'\cos(\theta - \alpha) + \cos(\theta - \beta).$$

If we eliminate β between these two, we get the required locus.

Subtracting one equation from the other, we have

$$\frac{l}{r} - \frac{l'}{r} = e\cos\theta - e'\cos(\theta - \alpha) \text{ .... (III)},$$

which is a straight line.

The perpendicular from the focus on the directrix of the first conic is, by the definition of a conic, $\frac{l}{e}$.

Hence the equation to the first directrix is

$$\frac{l}{r} = e\cos\theta \text{ .... (IV)},$$

and the second directrix
$$\frac{l'}{r} = e'\cos(\theta - \alpha) \text{ .... (V)};$$

and when these are simultaneously true, the equation (III) is satisfied; or, in other words, the locus goes through the intersection of the directrices.

Let ϕ and ϕ' be the angles made by (III) with (IV) and (V) respectively. Then, since (IV) is perpendicular to the initial line, it is evident that ϕ is the complement of the angle which (III) makes with the initial line.

Hence [by writing (III) in Cartesian co-ordinates] we have

$$\cot\phi = \frac{e - e'\cos\alpha}{e'\sin\alpha}, \text{ or } \sin\phi = \frac{e'\sin\alpha}{\sqrt{(e^2 + e'^2 - 2ee'\cos\alpha)}}.$$

Similarly $$\sin\phi' = \frac{e\sin\alpha}{\sqrt{(e^2 + e'^2 - 2ee'\cos\alpha)}};$$

$$\therefore \quad \sin\phi : \sin\phi' :: e' : e.$$

22. Let one ellipse be $a^2y^2 + b^2x^2 = a^2b^2$, and the other $a^2y^2 + b^2x^2 = pa^2b^2$; and let Q be on the former, and P, p on the latter.

Let the co-ordinates of Q be (h, k), and let θ be the inclination of the chord drawn from Q to cut the other ellipse, and r its length.

Then, by Art. 187,

$$r^2(a^2\sin^2\theta + b^2\cos^2\theta) + 2r(a^2k\sin\theta + b^2h\cos\theta) + a^2k^2 + b^2h^2 - pa^2b^2 = 0.$$

The two values of r will be PQ and Qp, and bearing in mind that they are of opposite sign we have

$$PQ \cdot Qp = \frac{pa^2b^2 - a^2k^2 - b^2h^2}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{(p-1)a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta},$$

since Q is on the first ellipse.

Hence $PQ \cdot Qp$ is constant, when θ is constant.

23. By Art. 293, the equation to any conic touching OX and OY at A and B is

$$\left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 + \mu xy = 0,$$

or $$\frac{x^2}{a^2} + \left(\frac{2}{ab} + \mu\right)xy + \frac{y^2}{b^2} - \frac{2x}{a} - \frac{2y}{b} + 1 = 0.$$

Hence, by Art. 270, the co-ordinates of the centre are given by

$$h = \frac{2a}{4 + ab\mu} \quad \text{and} \quad k = \frac{2b}{4 + ab\mu},$$

and therefore, whatever μ may be, the centre lies on the straight line

$$bx = ay.$$

24. Take the centre of the first two ellipses as origin, and let the axis of x bisect the angle between their major axes.

Hence the equation to one ellipse is

$$a^2(x\sin\alpha + y\cos\alpha)^2 + b^2(x\cos\alpha - y\sin\alpha)^2 = a^2b^2,$$

or $$x^2\left(\frac{\sin^2\alpha}{b^2} + \frac{\cos^2\alpha}{a^2}\right) + y^2\left(\frac{\cos^2\alpha}{b^2} + \frac{\sin^2\alpha}{a^2}\right)$$

$$+ xy\sin 2\alpha\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = 1 \quad \text{..................(I)},$$

and the equation to the other ellipse is

$$x^2\left(\frac{\sin^2\alpha}{b^2}+\frac{\cos^2\alpha}{a^2}\right)+y^2\left(\frac{\cos^2\alpha}{b^2}+\frac{\sin^2\alpha}{a^2}\right)$$
$$-xy\sin 2\alpha\left(\frac{1}{b^2}-\frac{1}{a^2}\right)=1 \quad\text{...................(II)}.$$

These may be abbreviated to

$$Px^2+Qy^2+Rxy=1 \quad\text{................................(III)},$$

and

$$Px^2+Qy^2-Rxy=1 \quad\text{................................(IV)}.$$

Now, since the circumscribing ellipse is evidently situated symmetrically with regard to the two others, its equation will be

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1 \quad\text{.......................................(V)}.$$

When (III) and (V) are simultaneously true, we have

$$\left(P-\frac{1}{A^2}\right)x^2+\left(Q-\frac{1}{B^2}\right)y^2+Rxy=0 \quad\text{...................(VI)}.$$

This equation represents two straight lines through the origin, and from the way it is obtained it must represent the common chords of the two ellipses concerned; but the two common chords are evidently coincident, since the curves *touch;* hence equation (VI) must be of the form $(y-mx)^2=0$. The condition for this, by Introd. § I., is

$$R^2=4\left(P-\frac{1}{A^2}\right)\left(Q-\frac{1}{B^2}\right).$$

Substituting the values of P, Q, R, we obtain the required condition.

Also, if the circumscribing ellipse is to be similar to the others, we must have $A^2 : B^2 :: a^2 : b^2$, and from this and the above condition we get the equation

$$A^4-A^2\left\{\frac{a^4\sin^2\alpha}{b^2}+b^2\sin^2\alpha+2a^2\cos^2\alpha\right\}+a^4=0.$$

Each of the roots of this equation will be real and positive, so that the value of A is always possible. Similarly for B; hence the circumscribing ellipse can always be similar.

25. Let the equation to one ellipse be

$$a^2y^2+b^2x^2=a^2b^2;$$

then the equation to the other is (by the preceding example)

$$x^2(a^2\sin^2\alpha+b^2\cos^2\alpha)+y^2(a^2\cos^2\alpha+b^2\sin^2\alpha)$$
$$+xy\sin 2\alpha(a^2-b^2)=n^2a^2b^2.$$

Multiply the first equation by n^2 and subtract the second from it, we get

$$x^2(n^2b^2-a^2\sin^2\alpha-b^2\cos^2\alpha)+y^2(n^2a^2-a^2\cos^2\alpha-b^2\sin^2\alpha)$$
$$-xy\sin 2\alpha(a^2-b^2)=0.$$

The same reasoning as in the previous example shews that this is a perfect square; hence

$$(a^2-b^2)^2 \sin^2\alpha \cos^2\alpha$$
$$=(n^2b^2-a^2\sin^2\alpha-b^2\cos^2\alpha)(n^2a^2-a^2\cos^2\alpha-b^2\sin^2\alpha),$$

or $$n^4a^2b^2-2a^2b^2n^2+2a^2b^2n^2\sin^2\alpha-a^4n^2\sin^2\alpha-b^4n^2\sin^2\alpha+a^2b^2=0;$$

$$\therefore\quad n^2\sin^2\alpha\,(a^2-b^2)^2=(n^2-1)^2a^2b^2;$$

$$\therefore\quad \sin\alpha=\left(n-\frac{1}{n}\right)\div\left(\frac{a}{b}-\frac{b}{a}\right)$$
$$=\left(n-\frac{1}{n}\right)\div\left(m-\frac{1}{m}\right).$$

26. Let the parabola be, by Art. 294,

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1.$$

And, by Ex. 21, Chap. VI., let the equation to the circle be

$$x+y-2\sqrt{(xy)}\sin\frac{\omega}{2}=c\cot\frac{\omega}{2}.$$

Solving these equations simultaneously, we get

$$y\left(a+b+2\sqrt{(ab)}\sin\frac{\omega}{2}\right)-2\sqrt{y}\left(a\sqrt{b}+b\sqrt{a}\sin\frac{\omega}{2}\right)+ab-bc\cot\frac{\omega}{2}=0.$$

This equation is to have *equal* roots, since the curves touch; hence, by Introd. § I., we have

$$\left(a\sqrt{b}+b\sqrt{a}\sin\frac{\omega}{2}\right)^2=\left(a+b+2\sqrt{(ab)}\sin\frac{\omega}{2}\right)\left(ab-bc\cot\frac{\omega}{2}\right).$$

Hence we get $$c\operatorname{cosec}\frac{\omega}{2}=\frac{ab}{(a+b)\sec\frac{\omega}{2}+2\sqrt{(ab)}\tan\frac{\omega}{2}}.$$

27. Let the curve be $$r=\frac{l}{1+e\cos\theta}.$$

Let one chord be drawn at an angle α; hence

$$r=\frac{l}{1+e\cos\alpha},\ \text{and}\ R=\frac{l}{1-e\cos\alpha};$$

$$\therefore\quad Rr=\frac{l^2}{1-e^2\cos^2\alpha}.$$

$$\therefore\quad \frac{1}{Rr}=\frac{1-e^2\cos^2\alpha}{l^2}.$$

Similarly $$\frac{1}{R'r'}=\frac{1-e^2\sin^2\alpha}{l^2};$$

$$\therefore\quad \frac{1}{Rr}+\frac{1}{R'r'}=\frac{2-e^2}{l^2},\ \text{a constant.}$$

28. In the first figure to Art. 146, let OP and Op be the fixed axes, and S the focus of the variable parabola. Take O as origin, and Op as the initial line.

Let $$pOS=\theta,\quad pOP=\alpha,\quad OS=r.$$

It is proved in Art. 146, that

$$pOS=POH=OPQ=OPS;$$

$$\therefore\quad OPS=\theta.$$

Similarly $$OpS=SOP=\alpha-\theta.$$

Also $$OSP=180^0-(SPO+SOP)=180^0-(SOp+SOP)=180^0-\alpha.$$

Similarly $$OSp=180^0-\alpha;$$

$$\therefore\quad Op=r\frac{\sin OSp}{\sin OpS}=r\frac{\sin\alpha}{\sin(\alpha-\theta)},$$

and $$OP=r\frac{\sin OSP}{\sin OPS}=r\frac{\sin\alpha}{\sin\theta}.$$

But, by hypothesis, $OP\,.\,Op=$ a constant $=m^2$ suppose;

$$\therefore\quad \frac{r^2\sin^2\alpha}{\sin\theta\,.\,\sin(\alpha-\theta)}=m^2,$$

or $$r=\frac{m}{\sin\alpha}\,.\,\sqrt{\{\sin\theta\,.\,\sin(\alpha-\theta)\}}.$$

29. [See fig. on next page.] Let the angle $SOP=\theta$; hence, by the preceding example,

$$SQO=\theta$$

$$\therefore\quad \frac{SZ}{SQ}=\sin\theta.$$

But, by Art. 141, $$SZ^2=a\,.\,SQ.$$

Divide one result by the other, and we get

$$SZ=\frac{a}{\sin\theta};$$

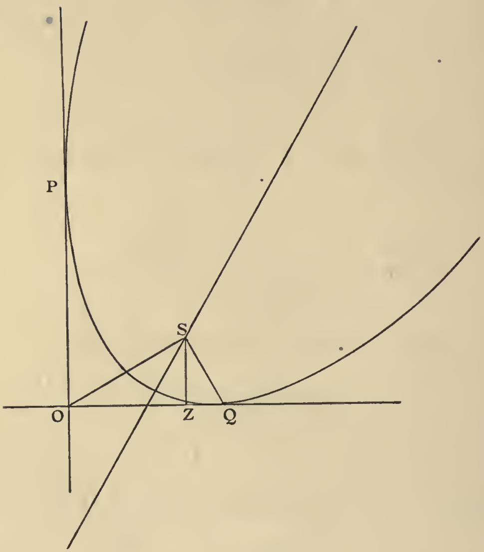
or, if (x, y) be the co-ordinates of S, we have

$$y=\frac{a}{\sin\theta}.$$

Similarly $$x=\frac{a}{\cos\theta}.$$

$$\therefore\quad x^2+y^2=\frac{a^2}{\sin^2\theta\,.\,\cos^2\theta}=\frac{4a^2}{(\sin 2\theta)^2};$$

$$\therefore \quad r = \frac{2a}{\sin 2\theta}.$$



30. In a parabola, the perpendicular from the vertex on a tangent making an angle θ with the axis is easily found to be

$$\frac{a \cos^2 \theta}{\sin \theta}.$$

Hence, if (x, y) be the co-ordinates of the vertex in this example, we have

$$y = \frac{a \cos^2 \theta}{\sin \theta}.$$

Similarly
$$x = \frac{a \sin^2 \theta}{\cos \theta};$$

$$\therefore \quad x^{\frac{4}{3}} y^{\frac{2}{3}} + x^{\frac{2}{3}} y^{\frac{4}{3}} = a^2 (\sin^2 \theta + \cos^2 \theta) = a^2,$$

which is the required locus.

31. Let the latus rectum be $2l$.

Let G_1, G_2 ...... be the successive centres, so that

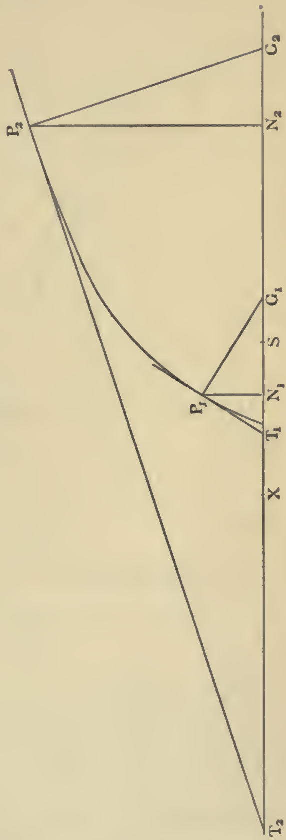
$$G_1G_2 = r_1 + r_2.$$

Let X be the foot of the directrix; therefore

$$P_2G_2^2 = P_2N_2^2 + N_2G_2^2$$
$$= 4a \,.\, AN_2 + 4a^2, \text{ by Art. 137}$$
$$= 4a \,.\, XN_2$$
$$= 2l \,.\, XN_2.$$

Similarly $P_1G_1^2 = 2l \,.\, XN_1$.

$$\therefore \quad r_2^2 - r_1^2 = 2l \,.\, N_1N_2 = 2l \,.\, G_1G_2$$
$$= 2l\,(r_1 + r_2);$$
$$\therefore \quad r_2 - r_1 = 2l.$$



32. Let CP and CQ be two perpendicular semi-diameters, so that PQ is one side of the parallelogram; and let CR be perpendicular to PQ; then it is required to prove that CR is constant.

Turn the axes round so that the equation becomes

$$Ax^2 + By^2 = n\,(a+b),$$

or

$$\frac{Ax^2}{n\,(a+b)} + \frac{By^2}{n\,(a+b)} = 1.$$

Hence, by Ex. 30, Chap. X.,

$$\frac{1}{CR^2} = \frac{A}{n\,(a+b)} + \frac{B}{n\,(a+b)} = \frac{A+B}{n\,(a+b)}.$$

But, by Art. 274,

$$A + B = a + b;$$
$$\therefore \quad CR^2 = n,$$

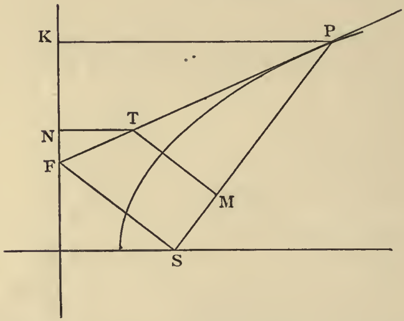
or

$$CR = \sqrt{n}, \text{ a constant.}$$

33. Draw PK perpendicular to directrix, and join SF; by Arts. 156, 207, SF is at right angles to SP.

Hence, by similar triangles,

$$SM : SP :: TF : PF$$
$$:: TN : PK;$$
$$\therefore\ TN : SM :: PK : SP$$
$$: 1 : e, \text{ by definition.}$$



34. Take the focus as origin; the equation to any of the conics can be expressed by $r = \dfrac{l}{1 + e\cos\theta}$.

Let α and β be the directions of the two chords; the corresponding values of r are $\dfrac{l}{1+e\cos\alpha}$ and $\dfrac{l}{1+e\cos\beta}$.

By Art. 58, the straight line joining these two points is

$$\frac{rl}{1+e\cos\alpha}\sin(\alpha-\theta) + \frac{l^2}{(1+e\cos\alpha)(1+e\cos\beta)}\sin(\beta-\alpha)$$
$$+\frac{rl}{1+e\cos\beta}\sin(\theta-\beta)=0.$$

This may be written

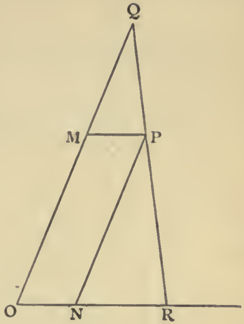
$$r[\sin(\alpha-\theta)+\sin(\theta-\beta)]+er[\cos\theta\,.\,\sin(\alpha-\beta)]-l\sin(\alpha-\beta)=0;$$

and whatever be the value of e this equation always gives the same value of r when $\theta=90^0$.

The value of r, in this case, is $\dfrac{l\sin(\alpha-\beta)}{\cos\beta-\cos\alpha}$.

It is evident that the point at which all the chords meet is on the common latus rectum.

35. Let P be the point dividing the moving line QR in the ratio of $a:b$; take the two fixed lines OR, OQ as axes.



Let $QOR=\omega$, $QRO=\theta$.

Let (x, y) be the co-ordinates of P.

Now $$\frac{x}{a}=\frac{PM}{QP}=\frac{\sin(\omega+\theta)}{\sin\omega};$$

$$\frac{y}{b}=\frac{PN}{PR}=\frac{\sin\theta}{\sin\omega};$$

$$\therefore \left(\frac{x}{a}-\frac{y\cos\omega}{b}\right)^2=1-\sin^2\theta=1-\frac{y^2\sin^2\omega}{b^2};$$

$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy\cos\omega}{ab}=1,$$

which is evidently an ellipse.

36. Let the conic be $$r=\frac{l}{1+e\cos\theta}.$$

Let the value r correspond to an angle α, and r' to $90^0+\alpha$;

$$\therefore r=\frac{l}{1+e\cos\alpha}, \text{ and } r'=\frac{l}{1-e\sin\alpha};$$

$$\therefore \frac{1}{r}-\frac{1}{l}=\frac{e\cos\alpha}{l}, \text{ and } \frac{1}{r'}-\frac{1}{l}=-\frac{e\sin\alpha}{l};$$

$$\therefore \left(\frac{1}{r}-\frac{1}{l}\right)^2+\left(\frac{1}{r'}-\frac{1}{l}\right)^2=\frac{e^2}{l^2}, \text{ which is constant.}$$

37. Let the conics be $r=\dfrac{l}{1+e\cos\theta}$, and $r=\dfrac{l}{1+e\cos(\theta-90^0)}$.

The tangent to the first at the point (α) is

$$\frac{l}{r}=e\cos\theta+\cos(\alpha-\theta).$$

The tangent to the second at the point $(\alpha+90^0)$ is

$$\frac{l}{r}=e\cos(\theta-90^0)+\cos(\alpha+90^0-\theta),$$

or $$\frac{l}{r}=e\sin\theta-\sin(\alpha-\theta).$$

Eliminating α we get $\left(\frac{l}{r}-e\cos\theta\right)^2+\left(\frac{l}{r}-e\sin\theta\right)^2=1$, which is a circle,

unless the conics are parabolas, when it becomes a straight line. Next if SPQ be a straight line, the tangent to the second conic at the point (α) is

$$\frac{l}{r}=e\sin\theta+\cos(\alpha-\theta).$$

Eliminating between the two tangents, we get $\cos\theta=\sin\theta$, or $\theta=45^0$, which is a straight line.

38. Let the semi-axes of the first ellipse be a, b, and the excentricity e; and c, d, e' be the corresponding quantities for the second ellipse;

$$\therefore\ c^2=SH^2+HL^2=4a^2e^2+a^2(1-e^2)^2=a^2(1+e^2)^2;$$

$$\therefore\ c=a(1+e^2).$$

Also $$d=HL=a(1-e^2).$$

$$\therefore\ e'=\frac{2e}{1+e^2}.$$

Draw PN the ordinate of P.

Let HP and QM meet at R; and let $HM=h$, $HN=x$.

$\therefore$ by similar triangles $SM:SN::SQ:SP$;

$\therefore$ $2ae+h:2ae+x::a(1+e^2)+\dfrac{2eh}{1+e^2}:a+e(ae+x)$ (see Art. 166).

From this we get, by reduction,

$$\frac{ah(1-e^2)}{x}-eh=a(1+e^2).$$

But $$HR:HP::HM:HN;$$

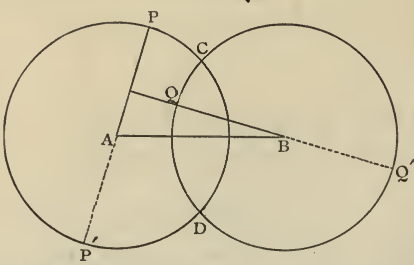
$$\therefore\ HR:a-e(ae+x)::h:x;$$

$$\therefore\ HR=\frac{ah(1-e^2)}{x}-eh=a(1+e^2).$$

Hence the locus of R is the auxiliary circle of the ellipse whose centre is H.

39. Let the radius be a;

$$\therefore\ AB=a\sqrt{2}.$$



Take A as origin, and let $PAB=\theta$.

The co-ordinates of C are $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$,

and those of D are $\left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$.

The co-ordinates of P are $(a \cos\theta, a \sin\theta)$, and those of Q are

$$(a\sqrt{2} - a\sin\theta, a\cos\theta).$$

Hence the equation of PQ is

$$y(\sqrt{2} - \sin\theta - \cos\theta) = x(\cos\theta - \sin\theta) + a\sqrt{2}\sin\theta - a,$$

and the co-ordinates of D satisfy this.

Similarly PQ', $P'Q$, $P'Q'$ will pass through C or D.

40. Since Q is intermediate to P and R, the three points P, Q, R are on the same branch of the curve if the curve is an hyperbola. Hence, whatever the curve is, we have (by Art. 288)

$$MSP = MSQ, \text{ or } MSQ = \tfrac{1}{2} PSQ.$$

Similarly $$QSN = \tfrac{1}{2} QSR;$$

$$\therefore MSN = \tfrac{1}{2} PSR.$$

41. Transferring the origin to the focus, the equation to the curve is $y^2 = 4a(x+a)$, and to the chord of contact is $ky = 2a(x+a+h)$.

The required equation is of the form $(y - mx)(y - nx) = 0$, and will therefore throughout be of the second order in x, y; hence from the above two equations we must obtain one of the second order in x, y.

From the second equation we have $\dfrac{ky - 2ax}{2a(a+h)} = 1$; hence the required equation is

$$y^2 = 4ax\left(\frac{ky - 2ax}{2a^2 + 2ah}\right) + 4a^2\left(\frac{ky - 2ax}{2a^2 + 2ah}\right)^2.$$

Transferring the origin back to the vertex, we get the result given.

42. Let one ellipse be $a^2y^2 + b^2x^2 = a^2b^2$; and the other

$$x^2(a^2\sin^2\theta + b^2\cos^2\theta) + y^2(a^2\cos^2\theta + b^2\sin^2\theta) + xy\sin 2\theta\,(a^2 - b^2) = a^2b^2.$$

Let the straight line $y = mx$ intersect the first ellipse;

$$\therefore x^2(a^2m^2 + b^2) = a^2b^2.$$

If it intersects the other ellipse, we have

$$x^2(a^2\sin^2\theta + b^2\cos^2\theta + a^2m^2\cos^2\theta + b^2m^2\sin^2\theta + ma^2\sin 2\theta - mb^2\sin 2\theta) = a^2b^2.$$

If the values of x from these two equations are identical (as will be the case for a common chord), we have

$$a^2m^2 + b^2 = a^2\sin^2\theta + b^2\cos^2\theta + a^2m^2\cos^2\theta + b^2m^2\sin^2\theta + ma^2\sin 2\theta - mb^2\sin 2\theta,$$

or $$m^2 - 2m\cot\theta - 1 = 0.$$

The values of m in this equation are such that $m_1 = -\dfrac{1}{m_2}$; in other words the two chords are at right angles.

43. Let OP and OQ be the two tangents, and let S be the focus. Take OP, OQ as axes, and let $OS=a$, $\angle SOP=\alpha$, $\angle SOQ=\beta$. On opposite sides of OS make the angles OSP, OSQ each $=\theta$; then (by Art. 288) PQ is a chord of contact.

Now $$OP=OS\cdot\frac{\sin OSP}{\sin OPS}=\frac{a\sin\theta}{\sin(\alpha+\theta)},$$

and $$OQ=OS\cdot\frac{\sin OSQ}{\sin OQS}=\frac{a\sin\theta}{\sin(\beta+\theta)}.$$

Hence the equation to PQ is

$$y\sin(\beta+\theta)+x\sin(\alpha+\theta)=a\sin\theta,$$

or $$\cos\theta\,(y\sin\beta+x\sin\alpha)=\sin\theta\,(a-y\cos\beta-x\cos\alpha).$$

Hence, whatever the value of θ is, this chord of contact always passes through the intersection of the two *fixed* straight lines $y\sin\beta+x\sin\alpha=0$,

and $$y\cos\beta+x\cos\alpha=a.$$

44. Let the equation to the ellipse be $a^2y^2+b^2x^2=a^2b^2$, and to the circle

$$x^2+y^2=b^2.$$

A tangent to the circle is $y=mx+b\sqrt{(1+m^2)}$ .... (A).

Also the polar of the point (h, k) with reference to the ellipse is

$$a^2ky+b^2hx=a^2b^2 \quad\text{.............................}\quad (B).$$

If (A) and (B) are identical we have

$$m=-\frac{b^2h}{a^2k},\ \text{and}\ b\sqrt{(1+m^2)}=\frac{b^2}{k}.$$

Eliminating m we get $a^4k^2+b^4h^2=a^4b^2$; hence the locus of (h, k) is an ellipse whose semi-axes are $\frac{a^2}{b}$ and b.

45. Take the focus as origin, and let the equation to the parabola be

$$y^2=4ax+4a^2.$$

Draw SR parallel to PQ.

Let the co-ordinates of P, Q, p, q, be respectively

$$(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3),\ (x_4, y_4).$$

The equation to PSp is of the form $y=mx$; solving this simultaneously with the equation to the curve we get

$$y^2-\frac{4ay}{m}-4a^2=0.$$

Hence (by Introd. § II.) $\quad y_1y_3=-4a^2$
Similarly $\quad y_2y_4=-4a^2$ .... (I).

Now the equation to PQ is $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)=\frac{4a}{y_2+y_1}(x-x_1)$,

or $$y(y_1+y_2)-4ax=y_1y_2+4a^2.$$

Hence the equation to SR is $y(y_1+y_2)-4ax=0$.

And the equation to pq is $y(y_3+y_4)-4ax=y_3y_4+4a^2$.

Solving these two together, and making use of the conditions (I) proved above, we get $x=-a$, which is the equation to the tangent at the vertex.

46. Let the chords be AP, AP', and let Q be the further angle; let AQ meet PP' in R.

Let the equation to AP be $y=mx$.

Hence the co-ordinates of P are $\left(\frac{4a}{m^2},\ \frac{4a}{m}\right)$.

Similarly the co-ordinates of P' are $(4am^2,\ -4am)$.

Hence, by Art. 10, the co-ordinates of R are

$$\left(\frac{2a}{m^2}+2am^2,\ \frac{2a}{m}-2am\right).$$

Therefore the co-ordinates of Q are $\left(\frac{4a}{m^2}+4am^2,\ \frac{4a}{m}-4am\right)$.

Let these be represented by (x, y);

$$\therefore\ y^2=\frac{16a^2}{m^2}+16a^2m^2-32a^2$$

$$=4ax-32a^2, \text{ which is a parabola.}$$

The vertex of this parabola is at a distance $8a$ from A.

47. Let QR intersect the normal at L. Taking P as origin, and axes parallel to the principal axes, the equation to the curve is (by Chap. IX. Ex. 48),

$$a^2y^2+b^2x^2+2a^2yk+2b^2xh=0.$$

Hence (by Art. 286) the co-ordinates of L are

$$-\frac{2b^2h}{a^2+b^2},\ -\frac{2a^2k}{a^2+b^2}.$$

Transfer the origin to the centre, and these co-ordinates become

$$h-\frac{2b^2h}{a^2+b^2},\ k-\frac{2a^2k}{a^2+b^2},$$

or $$\frac{a^2-b^2}{a^2+b^2}\cdot h,\ -\frac{a^2-b^2}{a^2+b^2}\cdot k.$$

Let these be represented by x and y; then we have

$$h=\frac{a^2+b^2}{a^2-b^2}\cdot x, \text{ and } k=-\frac{a^2+b^2}{a^2-b^2}\cdot y.$$

Substituting these in the equation $a^2k^2+b^2h^2=a^2b^2$, we get the required equation.

48. In the triangle ABC draw AO perpendicular to BC, and let OB, OA be the axes. Then the co-ordinates of A, B, C may be taken as $(0, a\sqrt{3})$, $(a, 0)$, $(-a, 0)$ respectively.

Let the equation to the equilateral hyperbola be (Art. 274),

$$Ax^2+Bxy-Ay^2+Dx+Ey+F=0.$$

By substituting successively the co-ordinates of the three angles of the triangle in this equation we find that it may be written

$$Ax^2+Bxy-Ay^2+\frac{4Aay}{\sqrt{3}}-Aa^2=0.$$

Hence the co-ordinates of the centre are $-\dfrac{4ABa}{\sqrt{3}(B^2+4A^2)}$, $\dfrac{8A^2a}{\sqrt{3}(B^2+4A^2)}$; and these are found to satisfy the equation $x^2+\left(y-\dfrac{a}{\sqrt{3}}\right)^2=\dfrac{a^2}{3}$, which is the equation to the inscribed circle.

49. Let the parabolas be $y^2=4ax$ and $y^2=-4ax$.

A tangent to the first is represented by $y=mx+\dfrac{a}{m}$.

Let this cut the second parabola in (x_1, y_1) and (x_2, y_2), and let (h, k) be the middle point of the chord, so that $h=\frac{1}{2}(x_1+x_2)$, $k=\frac{1}{2}(y_1+y_2)$.

Solving the equations to the tangent and to the second parabola simultaneously we have

$$y^2+\frac{4ay}{m}-\frac{4a^2}{m^2}=0.$$

$$\therefore\ y_1+y_2=-\frac{4a}{m}, \text{ and therefore } k=-\frac{2a}{m}.$$

Similarly $$h=-\frac{3a}{m^2}.$$

Hence (h, k) is on the parabola $y^2=-\dfrac{4ax}{3}$.

50. Let $OP=a$, $OQ=b$, and let S be the focus.

Draw SR parallel to OQ; and let $\angle SOR=\alpha$.

$$\therefore\ SOQ=\omega-\alpha.$$

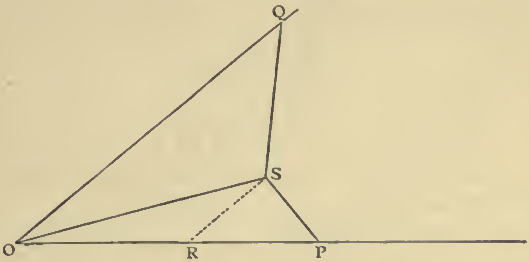
Also (by Chap. XIV. Ex. 28) $OSQ=OSP=180^0-\omega$; and $SQO=SOR=\alpha$.

Now $$\frac{OQ}{OS}=\frac{\sin\omega}{\sin\alpha}; \text{ and } \frac{OS}{OP}=\frac{\sin(\omega-\alpha)}{\sin\omega};$$

$$\therefore\ \frac{OQ}{OP} \text{ or } \frac{b}{a}=\frac{\sin(\omega-\alpha)}{\sin\alpha}.$$

Expanding we get $\sin^2 a = \dfrac{a^2 \sin^2 \omega}{a^2 + b^2 + 2ab \cos \omega}$.

Hence $\sin^2 (\omega - a) = \dfrac{b^2 \sin^2 \omega}{a^2 + b^2 + 2ab \cos \omega}$.



But $\dfrac{OR}{OS} = \dfrac{\sin (\omega - a)}{\sin \omega}$; and $\dfrac{OS}{OP} = \dfrac{\sin (\omega - a)}{\sin \omega}$;

$$\therefore \frac{OR}{a} = \frac{\sin^2 (\omega - a)}{\sin^2 \omega} = \frac{b^2}{a^2 + b^2 + 2ab \cos \omega}.$$

$$\therefore OR = \frac{ab^2}{a^2 + b^2 + 2ab \cos \omega}.$$

Similarly $SR = \dfrac{a^2 b}{a^2 + b^2 + 2ab \cos \omega}$.

51. Let the co-ordinates of the vertex be (h, k); then the equation to the axis is of the form $y - k = m(x - h)$ .... (I).

Now the axis is to cut the curve in only *one* finite point; solving (I) with the equation to the curve we get $(a + 2bm + cm^2) x^2 + \&c. = 0$. But this is to reduce to a *simple* equation, so that the coefficient of x^2 must be zero; hence

$$a + 2bm + cm^2 = 0.$$

From this, remembering that $b^2 = ac$ (by Art. 275), we have $m = -\dfrac{b}{c}$.

Also the tangent at the vertex (h, k) is

$$x(ah + bk + a') + y(bh + ck + c') + \&c. = 0;$$

and as this is to be perpendicular to the axis we have

$$\frac{ah + bk + a'}{bh + ck + c'} = \frac{1}{m} = -\frac{c}{b}.$$

Multiplying both sides by c, we have (since $b^2=ac$),

$$\frac{b(bh+ck)+a'c}{bh+ck+c'}=-\frac{c^2}{b},$$

$$\therefore\ bh+ck=-\frac{a'b+cc'}{a+c}.$$

But the equation to the axis is

$$\begin{aligned} y &= mx+k-mh \\ &= -\frac{bx}{c}+\frac{bh+ck}{c} \\ &= -\frac{bx}{c}-\frac{a'b+cc'}{ac+c^2}; \end{aligned}$$

and by means of the relation $b^2=ac$ this is easily seen to be identical with the given equation.

52. Fully worked in the Answers.

53. Fully worked in the Answers.

54. This can be worked by the method in the Answers, or as follows. (See figure to Art. 192.)

Let the normals intersect at Q_1, Q_2, Q_3, Q_4.

Let CE, CF be perpendicular to the normals at P and D;

$$\therefore\ Q_1Q_4=\frac{2CE}{\sin DCP};\ \ Q_1Q_2=\frac{2CF}{\sin DCP}.$$

But area of parallelogram

$$=Q_1Q_4\,.\,Q_1Q_2\,.\sin DCP=\frac{4CE\,.\,CF}{\sin DCP}.$$

And since (Chap. IX. Ex. 23) the normal at P is

$$ax\sec\phi-by\operatorname{cosec}\phi=a^2-b^2,$$

we have
$$CE=\frac{(a^2-b^2)\sin\phi\,.\cos\phi}{\sqrt{(a^2\sin^2\phi+b^2\cos^2\phi)}}.$$

Similarly
$$CF=\frac{(a^2-b^2)\sin\phi\,.\cos\phi}{\sqrt{(a^2\cos^2\phi+b^2\sin^2\phi)}},$$

and (Art. 41), $$\sin DCP=\frac{ab}{\sqrt{(a^2\sin^2\phi+b^2\cos^2\phi)}\sqrt{(a^2\cos^2\phi+b^2\sin^2\phi)}}.$$

Hence the result at once follows.

55. Taking the centre of the square as origin, and axes parallel to the sides, we get as the equation to the circle $x^2+y^2=2a^2$.

Also the equation $y^2-a^2=\lambda(x^2-a^2)$ evidently represents a curve going

through the angular points of the square, and by properly choosing λ it can represent any conic through these points.

The tangent to the circle at (x_1, y_1) is $xx_1+yy_1=2a^2$.

The tangent to the conic at (x', y') is (by Art. 283)

$$yy'-\lambda xx'=a^2(1-\lambda).$$

If these two tangents are identical we get, by equating like terms,

$$-\frac{x_1}{\lambda x'}=\frac{y_1}{y'}=\frac{2}{1-\lambda}.$$

But $$x_1^2+y_1^2=2a^2.$$

Hence, substituting, we get

$$2\lambda^2x'^2+2y'^2=a^2(1-\lambda)^2.$$

But $$(y'^2-a^2)=\lambda(x'^2-a^2).$$

Hence, eliminating λ we get the required locus.

56. Taking the figure to Art. 288, we see that, since the angle $PST=QST$, the perpendicular from T on $SP=$ that on SQ. Also since the angle which TQ makes with SQ is equal to the angle it makes with HQ produced, the perpendicular on $SQ=$ that on HQ. Hence the proposition is true.

57. This may be done as in the Answers, or as follows.

Let $R=$ required radius. Then, using the notation of Art. 288, we have

$$R^2=ST^2 \,.\, \sin^2 TSQ=ST^2 \,.\, \sin^2\frac{\beta-\alpha}{2}=ST^2-ST^2 \,.\, \cos^2\frac{\beta-\alpha}{2}.$$

Also by equation (1) of Art. 288,

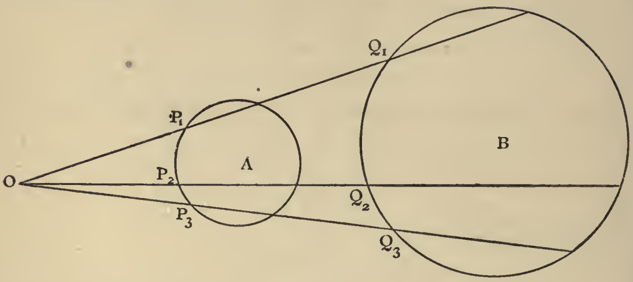
$$l=e \,.\, ST \,.\, \cos\frac{\alpha+\beta}{2}+ST \,.\, \cos\frac{\beta-\alpha}{2};$$

$$\therefore\ R^2=ST^2-\left(l-e \,.\, ST \,.\, \cos\frac{\alpha+\beta}{2}\right)^2.$$

Again, $$-\cos\frac{\alpha+\beta}{2}=\cos TSH=\frac{ae+x}{ST};$$

$$\begin{aligned}\therefore\ R^2&=ST^2-(l+ae^2+ex)^2\\&=(ae+x)^2+y^2-(a+ex)^2\\&=y^2+x^2(1-e^2)-a^2(1-e^2)\\&=\frac{a^2y^2+b^2x^2-a^2b^2}{a^2}.\end{aligned}$$

58. Since $OP_1 : OQ_1 :: OP_2 : OQ_2$;
$\therefore$ P_1P_2 is parallel to Q_1Q_2.



Similarly P_1P_3 and P_2P_3 are parallel to Q_1Q_3 and Q_2Q_3 respectively;

$$\therefore \text{ angle } P_1P_2P_3 = \text{angle } Q_1Q_2Q_3;$$

$\therefore$ angle $P_1AP_3 = Q_1BQ_3$, since these are doubles of the supplements of the preceding (Euclid III. 20, 22); $\therefore$ angle $AP_1P_3 = BQ_1Q_3$, and consequently AP_1 is parallel to BQ_1.

$\therefore$ $OA : OB :: OP_1 : OQ_1$, and therefore O is a centre of similitude (Art. 119, I.).

59. The tangents at (α) and (β) are (by Art. 288)

$$\frac{l}{r} = \cos\theta + \cos(\alpha - \theta) \text{ and } \frac{l}{r} = \cos\theta + \cos(\beta - \theta);$$

where these intersect we have $\cos(\alpha - \theta) = \cos(\beta - \theta)$, and therefore $\theta = \frac{\alpha + \beta}{2}$. Consequently $r = \frac{l}{2} \sec\frac{\alpha}{2} \cdot \sec\frac{\beta}{2}$. These values are easily seen to satisfy the given equation; and similarly for the other intersections.

Also, by Art. 105, it is evident that the given circle goes through the origin, that is to say, the focus.

60. The equation

$$4(ax^2 + bxy + cy^2 + dx + ey + f)(ah^2 + bhk + ck^2 + dh + ek + f)$$
$$= \{x(2ah + bk + d) + y(2ck + bh + e) + dh + ek + 2f\}^2$$

is evidently satisfied at the point (h, k).

Also, by Art. 285, it represents some locus going through the intersection of the curve and the chord of contact; hence it is only necessary to shew that it represents two straight lines.

Now, it will be found that the equation can be written in the form

$$(4ack^2 + 4aek + 4af - b^2k^2 - d^2 - 2bdk)(x - h)^2$$
$$+ (2b^2hk + 2bdh + 2bek + 4bf - 8achk - 4cdk - 4aeh - 2de)(x - h)(y - k)$$
$$+ (4ach^2 + 4cdh + 4cf - b^2h^2 - e^2 - 2beh)(y - k)^2 = 0;$$

hence it represents two straight lines.

CHAPTER XV.

1. Let $b=mb'$, so that $a-c=m(a'-c')$ or $a-ma'=c-mc'$.

Multiply the second equation by m and subtract from the first; we have

$$(a-ma')x^2+(c-mc')y^2+\&c.=0,$$

which is obviously a circle, passing through the intersections of the curves.

Again, let $f=nf'$; multiply the second equation by n and subtract it from the first; we have

$$(a-na')x^2+(b-nb')xy+(c-nc')y^2+(d-nd')x+(e-ne')y=0.$$

This goes through the origin, and through the intersections of the curves.

If this is a parabola we have (by Art. 275),

$$(b-nb')^2=4(a-na')(c-nc'),\text{ or}$$

$$n^2(b'^2-4a'c')+n(4a'c+4ac'-2bb')+b^2-4ac=0.$$

This equation will always give possible values for n unless

$$(2a'c+2ac'-bb')^2<(b'^2-4a'c')(b^2-4ac).$$

2. Take the line bisecting the angle between the two principal axes as axis of x. Now the equation to a conic referred to its principal axis as axis of x is of the form

$$ax^2+by^2+cx+e=0;$$

and if the axis be turned through 45^0 this becomes

$$x^2+y^2+P_1xy+Q_1x+R_1y+S_1=0.$$

Similarly the other conic will be denoted by

$$x^2+y^2+P_2xy+Q_2x+R_2y+S_2=0.$$

Multiply the first equation by P_2 and the second by P_1 and subtract, and we get the equation to a circle.

3. The equation to some locus through the origin and through the intersections of the given curves is

$$a'(Ay^2+2Bxy+Cx^2+2A'x)-A'(ay^2+2bxy+cx^2+2a'x)=0,$$

which reduces to

$$(a'A-aA')y^2+2(a'B-bA')xy+(a'C-cA')x^2=0.$$

This will represent two straight lines at right angles, if

$$a'A-aA'=-a'C+cA',\text{ or if }a'(A+C)=A'(a+c).$$

4. Take the asymptotes as axes, and let the equation to the hyperbola be $xy=m^2$.

Then, by Art. 337, the equation to the ellipse is $xy=w^2$, where $w=0$ is the equation to the chord of contact.

Combining these two equations we get $w=\pm m$, which represents two straight lines parallel to the chord of contact.

5. The equation $\alpha\beta=0$ represents the two straight lines $\alpha=0, \beta=0$; also it is evident that these lines can only meet the hyperbola at an infinite distance, and are consequently the asymptotes.

Also $\alpha^2-\beta^2=0$ represents the two straight lines $\alpha+\beta=0$ and $\alpha-\beta=0$, which bisect the angles between the asymptotes, and are therefore the axes.

Also $\alpha^2-n^2\beta^2=0$ represents the two straight lines

$$\alpha+n\beta=0 \text{ and } \alpha-n\beta=0.$$

Where $\alpha-n\beta=0$ meets the curve we have $\beta=\frac{c}{\sqrt{n}}$, $\alpha=c\sqrt{n}$. Also we shall find that the tangent at this point is $\alpha+n\beta-2c\sqrt{n}=0$, as may be seen by solving this equation simultaneously with the equation to the curve; if we do so we get two *coincident* values of $\beta=\frac{c}{\sqrt{n}}$.

But the tangent $\alpha+n\beta-2c\sqrt{n}=0$ is (by Art. 73) parallel to the diameter $\alpha+n\beta=0$, so that the diameters are conjugate.

6. Let the sides of the triangle be $\alpha=0, \beta=0, \gamma=0$; then the bisectors of the angles are $\beta-\gamma=0$, $\gamma-\alpha=0$, $\alpha-\beta=0$.

By Art. 318, the equation is of the form

$$\sqrt{(l\alpha)}+\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0.$$

When this meets $\alpha=0$, we have $\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0$,

or $$m\beta=n\gamma.$$

But, by hypothesis, $\beta=\gamma$ at this point;

hence $$m=n.$$

Similarly $$l=m=n,$$

so that the equation is $$\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}=0.$$

7. Let the parabola be $y^2=lx$, and let the two chords be drawn through the point (h, k), so that one chord is $y-k=\tan\alpha\,(x-h)$, and the other is

$$y-k=-\cot\alpha\,(x-h).$$

Now the equation $m(y^2-lx)=\{y-k-\tan\alpha\,(x-h)\}\,\{y-k+\cot\alpha\,(x-h)\}$ is satisfied at the points of intersection of the parabola with either chord. Also determining the value of m that this may be a parabola, we get by Art. 280, $$m=\operatorname{cosec}^2 2\alpha.$$

Hence the equation is of the form

$$x^2+y^2\cot^2 2\alpha-2xy\cot 2\alpha+Px+Qy+R=0,$$

and if the axes be turned through an angle 2α the terms containing x^2 and xy vanish, so that it becomes the equation to a parabola with its axis parallel to the new axis of x.

8. The given equation is satisfied by $y=k$, $x=h$. Also it is easily seen that the line $x=0$ cuts it in two coincident points, viz. (0, 0). Hence the curve touches the given parabola at the vertex. Similarly the curve is touched by $y=x+\frac{l}{4}$ at the point $\left(\frac{l}{4}, \frac{l}{2}\right)$, that is, it touches the parabola at the end of the latus rectum.

Also writing the equation in the form

$$y^2(lh-4hk+4h^2)+4x^2(lh-k^2)-4xy(lh-k^2)-lx(k-2h)^2=0,$$

it is, by Art. 280, an ellipse or hyperbola according as

$$(lh-k^2)^2-(lh-k^2)(lh-4hk+4h^2)$$

is negative or positive, or as $(k^2-lh)(k-2h)^2$ is negative or positive; that is, according as $k^2 \lesseqgtr lh$, or as (h, k) is inside or outside the parabola, by Art. 127.

9. Let the conic be

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta=0.$$

When $\alpha=0$ we have $m\beta=n\gamma$, which is therefore the equation to Aa.

Similarly the equations to Bb, Cc are $l\alpha=n\gamma$, and $l\alpha=m\beta$.

Taking the equation to Aa simultaneously with the equation to the conic we have $l\alpha(l\alpha-4m\beta)=0$. Hence for the point a' we have $l\alpha-4m\beta=0$; and as this equation is also true for the point C, it is the equation to Ca'.

Similarly the equation to Ac' is $n\gamma-4m\beta=0$, and consequently Ca' and Ac' intersect on the line $l\alpha=n\gamma$, that is to say, on Bb. Similarly for the others.

Again, the line $l\alpha+m\beta-n\gamma=0$ evidently goes through the intersection of $\alpha=0$ with $m\beta=n\gamma$, and of $\beta=0$ with $l\alpha=n\gamma$, so that it is the equation to ab.

Also the equation $l\alpha+m\beta-5n\gamma=0$ goes through the intersection of $l\alpha-4m\beta=0$ with $m\beta=n\gamma$, and through the intersection of $m\beta-4n\gamma=0$ with $l\alpha=n\gamma$, so that it is the equation to $a'b'$.

Hence ab and $a'b'$ evidently intersect on $\gamma=0$, that is to say, on AB.

10. Let the conic be $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$; the bisectors are $\beta=\gamma$, $\gamma=\alpha$, $\alpha=\beta$.

Where $\beta=\gamma$ meets the conic we get $l\beta+(m+n)\alpha=0$, and as this is satisfied at C, it is the equation to CA'.

Similarly, where $\beta=\gamma$ meets the conic we also get $l\gamma+(m+n)\alpha=0$, and as this is satisfied at B it is the equation to BA'.

Also the equation $(m+n)\alpha+(l+n)\beta-n\gamma=0$ goes through the intersection of $l\beta+(m+n)\alpha=0$ with $\beta=\gamma$, and through the intersection of

$$m\alpha+(l+n)\beta=0 \text{ with } \alpha=\gamma,$$

so that it is the equation to $A'B'$.

11. Using the notation of the previous question, the equation to AB is $\gamma=0$, and to $A'B'$ is

$$(m+n)\,\alpha+(l+n)\,\beta-n\gamma=0,$$

and these evidently intersect on the straight line

$$(m+n)\,\alpha+(l+n)\,\beta+(m+l)\,\gamma=0.$$

Similarly the other pairs intersect on the same line.

12. The equation is satisfied by $\frac{x}{a}+\frac{y}{b}-1=0$ simultaneously with $x=0$, and also by $\frac{x}{a}+\frac{y}{b}-1=0$ simultaneously with $y=0$; hence it is satisfied at the two points where $\frac{x}{a}+\frac{y}{b}-1=0$ cuts the axes. Similarly it is satisfied at the two points where $\frac{x}{a'}+\frac{y}{b'}-1=0$ cuts the axes. Hence it is a conic through these four points.

The co-ordinates of these four points are

$$(0,\ b),\quad (a,\ 0),\quad (0,\ b'),\quad (a',\ 0).$$

Also if the equation represents a parabola, we must have

$$\left(\mu+\frac{1}{ab'}+\frac{1}{a'b}\right)^2=\frac{4}{aa'bb'},$$

from which we get *two* values of μ; hence *two* parabolas can be drawn through four given points.

13. Let ABC be the original triangle, and let $A'B'C'$ be the other triangle, with A' on BC, &c.

Then since l, m, n are any constants, it is evident that the lines $B'C'$ and $B'A'$ can be represented by

$$u+nv+\frac{w}{m}=0,\ \text{ and }\ mu+\frac{v}{l}+w=0,$$

for these become identical when $v=0$, or in other words they intersect on AC.

Also the line $\frac{u}{n}+v+lw=0$ becomes identical with $B'C'$ when $w=0$, that is to say, it goes through C'. Similarly it goes through A'.

Hence it must be the line $A'C'$.

Also the line AA' is evidently $v+lw=0$,
and the line BB' is $w+mu=0$,
and the line CC' is $u+nv=0$.

If these are simultaneously true, we get by eliminating u, v, w, the condition

$$lmn=-1.$$

14. Let us take the diameter of the hyperbola which is a common chord as the axis of x, and its conjugate as axis of y. Let ω be the angle between the axes.

Let the equation to the hyperbola be $y^2 - x^2 = -k^2$, so that $(k, 0)$ and $(-k, 0)$ are two of the common points.

The equation to a circle through these two points is evidently of the form $$x^2 + y^2 + 2xy\cos\omega - 2By - k^2 = 0,$$
where B is an undetermined constant.

The centre of this circle, by Art. 104, is
$$\left(-\frac{B\cos\omega}{\sin^2\omega}, \frac{B}{\sin^2\omega}\right).$$

Adding the equations to the hyperbola and circle together, we get
$$2y(y + x\cos\omega - B) = 0.$$

This equation represents two straight lines, and it consequently represents two of the common chords: hence one common chord is
$$y + x\cos\omega - B = 0,$$
which is satisfied by the co-ordinates of the centre of the circle.

15. The co-ordinates of P are $(a\cos\theta, a\sin\theta)$; hence the equation to AP is $$y = \frac{\sin\theta}{\cos\theta - 1}(x - a),$$

and the equation to $A'P$ is
$$y = \frac{\sin\theta}{\cos\theta + 1}(x + a).$$

These can be combined in the one equation
$$x^2 - y^2 - a^2 - 2xy\cot\theta + 2ay\operatorname{cosec}\theta = 0,$$
or $$b^2x^2 - b^2y^2 - a^2b^2 - 2b^2xy\cot\theta + 2ab^2y\operatorname{cosec}\theta = 0.$$

Also the ellipse is $$b^2x^2 + a^2y^2 - a^2b^2 = 0.$$

To find the common chords of these two loci, subtract one equation from the other, and we get
$$y\{(a^2 + b^2)y + 2b^2x\cot\theta - 2ab^2\operatorname{cosec}\theta\} = 0.$$

As this represents two straight lines, it must be the equation to the two common chords AA' and QQ'; hence the equation to QQ' is
$$(a^2 + b^2)y + 2b^2x\cot\theta - 2ab^2\operatorname{cosec}\theta = 0,$$
which is equivalent to the given equation.

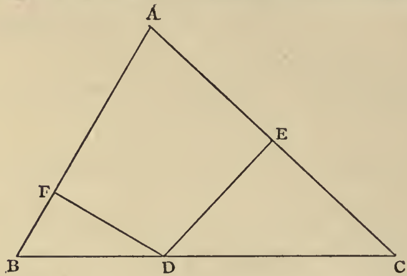
Again, the equation to the ordinate of P is $x = a\cos\theta$; hence, where this cuts QQ' we have
$$y = \frac{2ab^2}{a^2 + b^2}\sin\theta.$$

Eliminating θ between these two co-ordinates of R, we get
$$\left(\frac{x}{a}\right)^2 + \left\{\frac{y(a^2 + b^2)}{2ab^2}\right\}^2 = 1,$$
which is an ellipse.

16. Take the given triangle as the triangle of reference; then the equation to the locus is

$$\alpha^2+\beta^2+\gamma^2=k^2 \quad\text{.................................(I).}$$

If this is an ellipse its points of intersection with the line at infinity are



imaginary; solving the above equation simultaneously with $a\alpha+b\beta+c\gamma=0$, we get

$$\beta^2(a^2+b^2)+\gamma^2(a^2+c^2)+2bc\beta\gamma=k^2,$$

and, by Introd. § VI., the roots of this are imaginary; hence the curve is an ellipse.

Also, by Art. 73, the equation to the above conic can be written in the form

$$\alpha^2+\beta^2+\gamma^2=l^2(a\alpha+b\beta+c\gamma)^2;$$

where this intersects the line $\alpha=0$, we have

$$\beta^2(1-l^2b^2)+\gamma^2(1-l^2c^2)-2l^2bc\beta\gamma=0 \quad\text{..................(II).}$$

If the conic *touches* BC, this equation must have equal roots; hence, by Introd. § I., we have $\qquad l^2b^2+l^2c^2=1.$

Consequently equation (II) may be written

$$l^2c^2\beta^2+l^2b^2\gamma^2-2l^2bc\beta\gamma=0,$$

which reduces to $\qquad c\beta=b\gamma, \quad$ or $\dfrac{DE}{DF}=\dfrac{b}{c}$.

Hence $\qquad \dfrac{DC}{DB}=\dfrac{DE\operatorname{cosec}C}{DF\operatorname{cosec}B}=\dfrac{b\,.\,DE}{c\,.\,DF}=\dfrac{b^2}{c^2}.$

16. (*Aliter.*) Let the co-ordinates of the moving point be (h, k).

Take B as origin, and BC as axis of x.

The equation to AC is $\qquad y+(x-a)\tan C=0,$

and the equation to AB is $\qquad y-x\tan B=0.$

Hence the perpendiculars on the three sides are k, and

$$k\cos C+h\sin C-a\sin C, \quad\text{and}\quad k\cos B-h\sin B.$$

Hence, by hypothesis,

$$(k\cos C+h\sin C-a\sin C)^2+(k\cos B-h\sin B)^2+k^2=\text{constant}=m^2;$$

therefore (h, k) is on the curve

$$x^2(\sin^2 B+\sin^2 C)+y^2(\cos^2 B+\cos^2 C+1)+2xy(\sin C\cos C-\sin B\cos B)$$
$$-2ay\sin C\cos C-2ax\sin^2 C+a^2\sin^2 C-m^2=0.$$

This is easily seen to be an ellipse.

If the axis of x be a tangent, put $y=0$, and we get

$$x^2(\sin^2 B+\sin^2 C)-2ax\sin^2 C+a^2\sin^2 C-m^2=0,$$

and this is to have equal roots; hence

$$x=\frac{a\sin^2 C}{\sin^2 B+\sin^2 C}, \text{ or } BD=\frac{ac^2}{b^2+c^2}.$$

Also
$$CD=a-\frac{ac^2}{b^2+c^2}=\frac{ab^2}{b^2+c^2};$$

$$\therefore\ CD : BD :: b^2 : c^2.$$

17. Let O be the centre of the circle, and OE, OF the perpendiculars on BC and AC. Then the equation to OC is evidently

$$\frac{\alpha}{\beta}=\frac{OE}{OF}=\frac{OC\,.\cos A}{OC\,.\cos B}=\frac{\cos A}{\cos B}.$$

18. The bisector of the angle C evidently passes through D.

Also the equation $\alpha\sin C+\gamma(\sin A+\sin B)=0$ is satisfied when $\alpha=\beta$ is true simultaneously with $\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C=0$, that is to say, it passes through the intersection of the circle and the bisector of C; hence it is a line through D. Moreover from its form it is a line through B; hence it is the line BD.

Similarly for the other one.

19. In the equation of Art. 318, put $u=0$, and we have

$$m^2v^2+n^2w^2-2mnvw=0,$$

or
$$mv-nw=0,$$

which is consequently the equation to AA'.

Similarly the equation to BB' is $lu-nw=0$.

Now the given equation is satisfied at the intersection of $mv-nw=0$ with $u=0$, and also at the intersection of $lu-nw=0$ with $v=0$; hence it is the line $A'B'$.

20. The equation to the conic may be written

$$(nw+mv)(nw-mv)=l^2u^2;$$

hence by Art. 337, $nw+mv=0$ and $nw-mv=0$ are the tangents for which $u=0$ is the corresponding chord of contact; in other words they are the tangents at A and C.

Let BD meet FE in L, then, by Art. 339, LA and LC are the tangents at A and C.

Also, writing the equation in the shape $(nw+lu)(nw-lu)=m^2v^2$, we find the tangents at B and D to be

$$nw+lu=0, \quad \text{and} \quad nw-lu=0.$$

Again, since AB goes through the intersection of $u=0$ with $nw+mv=0$, and also through the intersection of $v=0$ with $nw+lu=0$, its equation must be
$$lu+mv+nw=0.$$

Similarly, BC is $lu-mv+nw=0$; CD is $lu+mv-nw=0$; and DA is
$$lu-mv-nw=0.$$

Also FG goes through the intersection of $lu-mv+nw=0$ with $w=0$, and through the intersection of $v=0$ with $u=0$; hence its equation is $lu-mv=0$; and this is evidently satisfied when $nw+mv=0$ and $nw+lu=0$ simultaneously, and also when $nw-mv=0$ and $nw-lu=0$ simultaneously.

21. The quantity α in Art. 323 can be replaced (see Art. 47) by

$$\frac{y-m_1x-\dfrac{a}{m_1}}{\sqrt{(1+m_1^2)}}.$$

Also, by Art. 41,

$$\sin C=\frac{m_2\sim m_1}{\sqrt{(1+m_2^2)}\sqrt{(1+m_1^2)}}.$$

Hence the equation to the circle becomes

$$\frac{y-m_1x-\dfrac{a}{m_1}}{\sqrt{(1+m_1^2)}}\times\frac{y-m_2x-\dfrac{a}{m_2}}{\sqrt{(1+m_2^2)}}\times\frac{m_2\sim m_1}{\sqrt{(1+m_2^2)}\sqrt{(1+m_1^2)}}+\&c.=0.$$

This equation reduces to

$$(1+m_3^2)(m_2\sim m_1)\left(y-m_1x-\frac{a}{m_1}\right)\left(y-m_2x-\frac{a}{m_2}\right)+\&c.=0.$$

Now the given equations represent three tangents to the parabola $y^2=4ax$, and the co-ordinates of the focus are $(a, 0)$. It is easily seen that these co-ordinates satisfy the equation to the circle.

22. Let the triangle ABC have its sides represented by $u=0$, $v=0$, $w=0$; then by Art. 110, $lv+mu=0$ is the tangent at C.

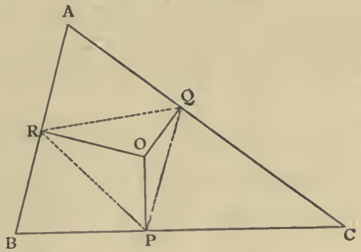
Also if from any point on the conic a perpendicular be drawn to $lv+mu=0$, then (by Art. 78, v.) this perpendicular is proportional to the particular value assumed by $lv+mu$ when the co-ordinates of the point are inserted. Hence the equation $w(lv+mu)=-nuv$ implies, that if from any point on a conic circumscribing ABC perpendiculars are drawn to the sides and to the tangent at C, the product of the perpendiculars on this tangent and on AB bears a constant ratio to the product of the perpendiculars on AC and BC. Also this is evidently what the theorem in Art. 336 becomes when two angles of the quadrilateral coincide, so that the side joining them becomes a tangent.

23. The equation may be written

$$\alpha(\gamma \sin B + \beta \sin C) = -\beta\gamma \sin A.$$

But $\gamma \sin B + \beta \sin C = 0$ is the tangent at A to the circumscribed circle (Art. 310).

Hence the above equation is equivalent to the following theorem: if from any point on the circumscribed circle perpendiculars are drawn to the sides of the triangle and to the tangent at one angle, then the product of the perpendiculars on the tangent and opposite side bears a constant ratio to the product of the perpendiculars on the other two sides.



Again, if from any point O we draw perpendiculars to the sides of ABC, then $\beta\gamma \sin A = OQ \cdot OR \sin ROQ =$ twice area of OQR.

Similarly, $\alpha\beta \sin C =$ twice area of POQ,

and $\gamma\alpha \sin B =$ twice area of POR.

Hence $\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C =$ twice area of PQR; consequently the equation of Art. 323 asserts that if O is on the circumscribing circle the area of PQR is zero, or P, Q, R are in one straight line.

24. Fully worked in the Answers.

25. By Art. 337, the three conics are $\beta\gamma - \alpha^2 = 0$, $\gamma\alpha - \beta^2 = 0$, $\alpha\beta - \gamma^2 = 0$.

Also the equations to the lines joining the angles to the centre of inscribed circle are (Art. 72),

$$\beta - \gamma = 0, \ \gamma - \alpha = 0, \ \alpha - \beta = 0.$$

Now since the first equation may be written $\beta(\beta + \gamma - 2\alpha) - (\alpha - \beta)^2 = 0$, it is the equation to a conic touching the two lines $\beta = 0$ and $\beta + \gamma - 2\alpha = 0$ at the points where they meet $\alpha - \beta = 0$.

Hence the tangent to the first conic at the centre of inscribed circle is

$$\beta + \gamma - 2\alpha = 0.$$

This line intersects $\alpha = 0$ on the line $\alpha + \beta + \gamma = 0$; and so for the others.

Again, the first equation may be written $\beta(\gamma+4\alpha+4\beta)-(\alpha+2\beta)^2$, and the second may be written $\alpha(\gamma+4\alpha+4\beta)-(\beta+2\alpha)^2$; hence both conics touch the line
$$\gamma+4\alpha+4\beta=0.$$

Also this common tangent evidently intersects the line $\gamma=0$ on the before-mentioned line
$$\alpha+\beta+\gamma=0.$$

26. (See Example 5 of this chapter.)

Let one hyperbola be $\beta\gamma=c^2$; then since its vertex is on the line $\beta=\gamma$, it follows that the perpendicular from A' on AB or AC is c;
$$\therefore\ c=AA'.\sin\tfrac{1}{2}A.$$

Hence, by hypothesis, c is to be the same for each hyperbola, or in other words the hyperbolas are $\beta\gamma=c^2$, $\gamma\alpha=c^2$, $\alpha\beta=c^2$.

Where the first intersects the second we get $\alpha=\beta$, and this is the equation to the axis of the third one.

27. If for α we put $x\cos\alpha+y\sin\alpha-p$ (Art. 71), and similarly for β and γ we get
$$x^2(l\cos^2\alpha+m\cos^2\beta+n\cos^2\gamma)+y^2(l\sin^2\alpha+m\sin^2\beta+n\sin^2\gamma)+\&c.=0;$$
and by Art. 274, the sum of the coefficients of x^2 and y^2 must be zero, or
$$l+m+n=0.$$

In the second instance the equation becomes
$$x^2(l\cos\beta\cos\gamma+m\cos\gamma\cos\alpha+n\cos\alpha\cos\beta)$$
$$+y^2(l\sin\beta\sin\gamma+m\sin\gamma\sin\alpha+n\sin\alpha\sin\beta)+\&c.=0;$$
and consequently the condition is
$$l\cos(\beta-\gamma)+m\cos(\gamma-\alpha)+n\cos(\alpha-\beta)=0, \text{ or, by Art. 78, I.,}$$
$$l\cos A+m\cos B+n\cos C=0.$$

28. This may be done according to the method indicated in the Answers, or as follows:

The tangent to a parabola at (x', y') is of the form $y=\frac{2a}{y'}x+\frac{y'}{2}$, which becomes $y=\infty$ when x' and y' are infinite. Hence the line at infinity is a tangent to any parabola, or in other words it cuts the parabola in two *coincident* points. The tangent to an hyperbola is of the form $y=mx-\frac{b^2}{y'}$, and when x', y' are infinite this takes the form $y=mx$, which is a line through the origin. Hence the line at infinity is not a tangent to an hyperbola, or in other words it cuts the hyperbola in two *non-coincident* points.

Also as the ellipse is a closed curve the line at infinity cuts it in two *imaginary* points.

Let us therefore solve the equations $\sqrt{(l\alpha)}+\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0$ and $a\alpha+b\beta+c\gamma=0$ simultaneously.

Eliminating a we get

$$\beta(bl+am)+\gamma(cl+an)+2a\sqrt{(mn\beta\gamma)}=0 \;\ldots\ldots\ldots\ldots\ldots\; \text{(I)}.$$

If the curve is an ellipse, the roots of this are imaginary, in which case

$$a^2mn<(bl+am)(cl+an),$$

or

$$l\left\{\frac{l}{a}+\frac{m}{b}+\frac{n}{c}\right\}>0.$$

But it may be easily seen (as in Art. 324) that, in the case of the ellipse, all the quantities l, m, n are of the same sign. Hence the condition just obtained is equivalent to $lmn\left(\frac{l}{a}+\frac{m}{b}+\frac{n}{c}\right)>0.$

In the case of the parabola, the two roots of the equation (I) are coincident, so that we get

$$lmn\left(\frac{l}{a}+\frac{m}{b}+\frac{n}{c}\right)=0.$$

In the case of the hyperbola it is easily seen that two of the quantities l, m, n are positive and one negative, so that the condition is

$$lmn\left(\frac{l}{a}+\frac{m}{b}+\frac{n}{c}\right)<0.$$

29. Using the method of the preceding, and solving the given equation with $a\alpha+b\beta+c\gamma=0$, we get $bn\beta^2+cm\gamma^2+(cn-al+bm)\beta\gamma=0$; and the roots of this are imaginary, coincident, or non-coincident, according as

$$a^2l^2+b^2m^2+c^2n^2-2lmab-2mnbc-2nlca$$

is negative, zero or positive.

30. By Art. 335, our required equation is $S=l(a\alpha+b\beta+c\gamma)^2$; and by Art. 334,

$$S=\alpha^2\cos^4\tfrac{1}{2}A+\&c.\ldots$$

But our circle is to be satisfied by the co-ordinates of A, viz.

$$\alpha=c\sin B,\ \beta=0,\ \gamma=0.$$

Hence we have $c^2\sin^2 B\,.\cos^4\tfrac{1}{2}A=la^2c^2\sin^2 B,$

or

$$l=\frac{\cos^4\frac{1}{2}A}{a^2}.$$

Hence our equation is determined.

31. Writing the equations in the shape

$$w(lv+mu)=-nuv, \text{ and } w(l'v+m'u)=-n'uv,$$

and dividing one by the other, we get

$$\frac{lv+mu}{l'v+m'u}=\frac{n}{n'}, \text{ or } u(mn'-m'n)=v(l'n-ln').$$

Similarly by eliminating v we get $u(mn'-m'n)=w(lm'-l'm)$, and these two conditions determine the required point.

32. Let $S_1=0$, $S_2=0$, be the equations to two circles, then their radical axis will be $S_1-k\,S_2=0$, if k be so chosen as to make the terms containing x^2 and y^2 vanish.

Hence if the equation to the radical axis is $l\alpha+m\beta+n\gamma=0$, we have $S_1-kS_2=(a\alpha+b\beta+c\gamma)(l\alpha+m\beta+n\gamma)$, since $a\alpha+b\beta+c\gamma$ is constant.

Using the value of S_1 given in Art. 334, and that of S_2 in Art. 323, we get

$$\alpha^2\cos^4\frac{A}{2}+\beta^2\cos^4\frac{B}{2}+\gamma^2\cos^4\frac{C}{2}-2\beta\gamma\cos^2\frac{B}{2}\cos^2\frac{C}{2}-2\gamma\alpha\cos^2\frac{C}{2}\,.\cos^2\frac{A}{2}$$

$$-2\alpha\beta\cos^2\frac{A}{2}\,.\cos^2\frac{B}{2}-k\,(\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C)$$

$$=(a\alpha+b\beta+c\gamma)(l\alpha+m\beta+n\gamma).$$

Equating the coefficients of α^2, β^2, γ^2 we get

$$l:m:n::\operatorname{cosec}A\,.\cos^4\frac{A}{2}:\operatorname{cosec}B\,.\cos^4\frac{B}{2}:\operatorname{cosec}C\,.\cos^4\frac{C}{2}.$$

33. The equation $$\frac{n\beta+m\gamma}{a}=\frac{l\gamma+n\alpha}{b}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;\text{(I)},$$

is satisfied when $n\beta+m\gamma=0$, simultaneously with $l\gamma+n\alpha=0$, or in other words it is satisfied at the intersection of the tangents at A and B. Also it is satisfied by $\gamma=0$ simultaneously with $\dfrac{\alpha}{\beta}=\dfrac{b}{a}$, that is to say it is satisfied at the middle point of AB; hence it is a diameter. Similarly

$$\frac{l\gamma+n\alpha}{b}=\frac{m\alpha+l\beta}{c}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;\text{(II)},$$

is a diameter.

Now the required diameter is to pass through the centre, that is through the intersection of (I) and (II), and through the intersection of $\beta=0$ with $\gamma=0$. Hence the desired equation is obtained by eliminating α between equations (I) and (II).

34. The tangent is to be of the form $\gamma=$ a constant, or $k\gamma=a\alpha+b\beta+c\gamma$.

And, by Art. 322, the condition that this may be a tangent is

$$\frac{l}{a}+\frac{m}{b}+\frac{n}{c-k}=0.$$

Determining k from this, we get as the equation to the tangent

$$(nab+lbc+mac)\,\gamma=2\Delta\,(lb+ma).$$

But the line parallel to this and midway between it and $\gamma=0$ must contain the centre. Hence the centre is on the line

$$(nab+lbc+mac)\,\gamma=\Delta\,(lb+ma),$$

or $$\frac{\gamma}{lb+ma}=\frac{\Delta}{nab+lbc+mac}.$$

Similarly, at the centre, $\dfrac{\alpha}{mc+nb}$ and $\dfrac{\beta}{na+lc}$ are equal to the same quantity.

35. Taking the transformed equation of Art. 332, and comparing it with the equation in Example 29, we see that $l=2L'-\frac{Mc}{b}-\frac{Nb}{c}$, and

$$m=2M'-\frac{Na}{c}-\frac{Lc}{a}, \text{ and } n=2N'-\frac{Lb}{a}-\frac{Ma}{b}.$$

Substituting these values in the condition of Example 29, we get that the required condition is, according as

$$a^2(L'^2-MN)+b^2(M'^2-LN)+c^2(N'^2-LM)+2bc(LL'-M'N')$$
$$+2ac(MM'-L'N')+2ab(NN'-L'M')$$

is negative, zero, or positive.

36. The equation to the conic is of the form

$$l\beta\gamma+m\gamma\alpha+n\alpha\beta=0.$$

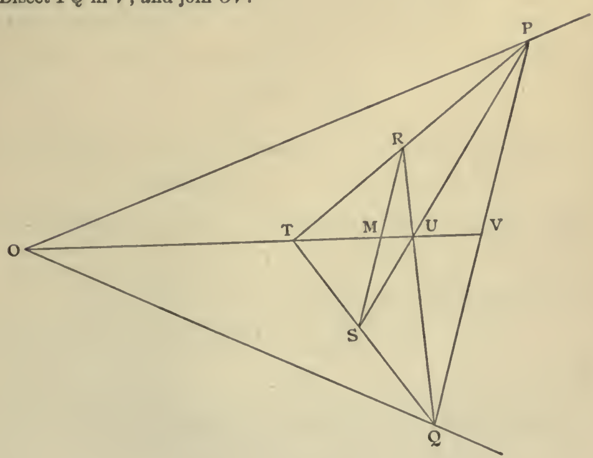
Here the tangent at A is $m\gamma+n\beta=0$. Any line parallel to this is represented by $m\gamma+n\beta+k(a\alpha+b\beta+c\gamma)=0$, and if this is equivalent to $\alpha=0$, we get $$m+kc=0,\ n+kb=0.$$

Hence $$\frac{m}{n}=\frac{c}{b}, \text{ or } m:n::\frac{1}{b}:\frac{1}{c}.$$

Hence the conic is $\frac{\beta\gamma}{a}+\frac{\gamma\alpha}{b}+\frac{\alpha\beta}{c}=0$, and by the condition in Example 29, this is an ellipse.

37. See Example 4, of this chapter.

Bisect PQ in V, and join OV.



Then OV is a diameter of the ellipse (Art. 201), and consequently it bisects RS. Let M be the middle point of RS, and let PR meet OV in T.

Then $$TM : TV :: MR : PV :: MS : QV,$$
hence SQ also meets OV in T.

Now TV is a diameter of the hyperbola, bisecting RS, and therefore if TR is the tangent at R, TS will be the tangent at S (Art. 253).

Also if PS cuts OV at U, we have
$$MU : UV :: MS : PV :: MR : QV,$$
hence U is also the point where QR cuts OV.

38. Let (t', u', v', w') and (t'', u'', v'', w'') be two points on the conic; then, bearing in mind that $t'u'=v'w'$ and $t''u''=v''w''$, it is easily seen that the straight line $(t-t')\,u''-(v-v')\,w''=v'w-t'u$ is satisfied by the co-ordinates of both points, and is consequently the chord joining them. Make the two points coincide, and the chord becomes a tangent and its equation takes the required form.

39. Taking $\alpha=0$, $\beta=0$, $\gamma=0$, as the sides of the triangle of reference, the conic may be represented by $\sqrt{(l\alpha)}+\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0$. By Art. 322, the condition that $\frac{\alpha}{a_1}+\frac{\beta}{b_1}+\frac{\gamma}{c_1}=0$ should touch the conic is $la_1+mb_1+nc_1=0$. Similarly, $la_2+mb_2+nc_2=0$, and $la_3+mb_3+nc_3=0$. Eliminating l, m, n, we get the required condition.

40. It is evident that DE, EF, DF, are parallel to the sides of the triangle ABC. Consequently the bisector of FDE is parallel to the bisector of A. Hence its equation is of the form $\beta-\gamma=k$. This equation is to be satisfied by the co-ordinates of D, viz. $\beta=\frac{a}{2}\sin C$, $\gamma=\frac{a}{2}\sin B$. Hence
$$k=\frac{a}{2}(\sin C-\sin B),$$
and our required equation is therefore $\beta-\gamma=\frac{a}{2}(\sin C-\sin B)$. This is easily shewn to be equivalent to the result in the Answers.

41. Let the equation be $\sqrt{(l\alpha)}+\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0$. Where this touches the side $\alpha=0$, we have $\sqrt{(m\beta)}+\sqrt{(n\gamma)}=0$, or $\frac{\beta}{\gamma}=\frac{n}{m}$.

But since it touches at the middle point of the side
$$\frac{\beta}{\gamma}=\frac{\frac{1}{2}a\sin C}{\frac{1}{2}a\sin B}=\frac{c}{b}; \quad \therefore \frac{n}{m}=\frac{c}{b}.$$

Similarly $$\frac{l}{n}=\frac{a}{c}.$$

Hence the equation is $\sqrt{(a\alpha)}+\sqrt{(b\beta)}+\sqrt{(c\gamma)}=0$.

By the test in Example 28 this is seen to be an ellipse.

42. Let the chord through Q meet the conic at A and B.

Let the equation to the conic be $uv - w^2 = 0$. Then the equation to PQ is $u - v = 0$. The equation $u - v - kw = 0$ is evidently satisfied at the intersection of $u - v = 0$ with $w = 0$, and therefore passes through Q, and by varying k it may be the equation to any chord AB through Q. Also the equation $(u-v)^2 = k^2uv$ is evidently the equation to two straight lines, and it is satisfied at the intersection of $u = 0$ and $v = 0$, that is, at P, and also at the intersection of $u - v - kw = 0$ with $uv - w^2 = 0$; hence it represents the two lines PA, PB. Also since this equation is of the form $(u - lv)(lu - v) = 0$, it represents two straight lines equally inclined to $u - v = 0$.

CHAPTER XVI.

1. It is evident that each major axis has its ends on the same two generating lines of the cone; hence the problem is a particular case of that solved in Chapter XIV., Example 35.

2. This is demonstrated in the Answers. For a fuller investigation reference may be made to Drew's *Geometrical Conics*, Chapter IV.

3. Using the figure and notation of Art. 344, the equation to any parabolic section is $y^2 = \dfrac{2cx \sin\theta \,.\, \sin\alpha}{\cos\alpha}$, or $y^2 = 4cx \,.\, \sin^2\alpha$, since $\theta = \pi - 2\alpha$.

Hence, if S is the focus $\qquad AS = c\sin^2\alpha$.

Let the angle $SOA = \phi$, so that $ASO = 2\alpha - \phi$.

Now
$$\frac{\sin ASO}{\sin SOA} = \frac{c}{AS},$$
or
$$\frac{\sin(2\alpha - \phi)}{\sin\phi} = \operatorname{cosec}^2\alpha.$$

Hence $\cot\phi = \dfrac{\cos 2\alpha + \operatorname{cosec}^2\alpha}{\sin 2\alpha}$, whence we see that ϕ is constant, and therefore so is the angle that SO makes with OH. Hence as the plane of section varies, OS will describe a right circular cone round OH as axis.

4. In the same manner as in Art. 345, if e be the excentricity of the given hyperbola, we have the condition that $e^2\cos^2\alpha$ must not be >1, or e^2 not $>\sec^2\alpha$. If ϕ be the semi-angle between the asymptotes of the given hyperbola, this condition becomes $1 + \tan^2\phi$ not $> 1 + \tan^2\alpha$, or ϕ not $> \alpha$.

5. From the results of Art. 344, it is easily seen that the Latus Rectum of the section is $2c \,.\, \sin\theta \,.\, \tan\alpha$; also the perpendicular from the vertex on the plane of section is $c \,.\, \sin\theta$; and these two are evidently in a constant ratio.

6. Let ABC be the triangle, and $\alpha=0$, $\beta=0$, $\gamma=0$, its sides. The perpendicular from C on AB is evidently represented by $\alpha\cos A-\beta\cos B=0$. Also the sides meeting at C are $\alpha=0$, $\beta=0$; hence the remaining ray of the harmonic pencil is (by Art. 356) $\alpha\cos A+\beta\cos B=0$; this is therefore the equation to the tangent at C. Similarly the other tangents are

$$\alpha\cos A+\gamma\cos C=0, \text{ and } \beta\cos B+\gamma\cos C=0.$$

Hence (Arts. 309, 310), the conic is $\beta\gamma\sec A+\gamma\alpha\sec B+\alpha\beta\sec C=0$.

7. See the figure to Art. 75. Let $u=0$, $v=0$, $w=0$, be the equations to AC, BD, EF respectively.

Now the equations $lu+mv+nw=0$, and $lu+mv-nw=0$, can represent any lines whatever meeting on $w=0$; hence they may be taken for the equations to BC and AD. Since CD goes through the intersection of $lu+mv+nw=0$ with $u=0$, and through the intersection of $lu+mv-nw=0$ with $v=0$, its equation must be $-lu+mv+nw=0$.

Similarly the equation to AB must be $lu-mv+nw=0$.

8. In the figure of Art. 360, let Nl cut AB at G; join Gn and Gl. Then GO, GN, GM, GL is an harmonic pencil, and so is GO, Gn, Gm, Gl; but as GO is common to both pencils, and Gl, Gm, are the produced parts of GN, GM, it follows (by Art. 354), that Gn is the produced part of GL.

This might also be done as follows: by Art. 292, the line joining the intersection of Nn and Ll to the intersection of Nl and Ln is the polar of O; hence the intersection of Nl and Ln is on the polar of O, that is, on AB.

CHAPTER XVII.

1. Let CP, CD be two radii of a circle at right angles, and let the tangents at P, D meet at T. Then it is evident that the triangles CPT, CDT are equal; consequently they will be equal when the circle is projected into an ellipse.

2. Take CP, CD as before, and K any point in PD; and CL parallel to PD. Then the area of $CLK=\frac{1}{2}CL\times$ perpendicular from C on PD; hence the area is independent of the position of K. Then project, &c.

3. Take CP, CD as before, and PQ parallel to any fixed line. Then (by Euclid III. 22), the angle $PQD=135^0$, and therefore DQ is parallel to a fixed line inclined at 135^0 to the former fixed line. Then project, &c.

4. Let CA, CB be two radii at right angles, and AP, BQ be drawn parallel to one another to meet the circle in P, Q. Then the angle $APQ=BQP$, $\therefore$ arc $AQ=BP$; $\therefore$ since BA is a quadrant, PQ is a quadrant, and CP, CQ are radii at right angles. Hence they will project into conjugate diameters.

5. In a circle let CP, CQ be any radii, and let the tangent at Q meet CP in N, and the tangent at P meet CQ in M. Then it is evident that the triangles CPM, CQN are equal. If now the circle be projected into an ellipse, PM will be parallel to the conjugate of CP, &c.

6. Let PQ be a diameter of a circle, and R, S any points on the circumference. Let PR and QS meet at M, and QR and PS at T; also let RS meet PQ in L. Then $PRSQ$ is a quadrilateral in a circle, and therefore (by Art. 292), MT is the polar of L; but the polar of L is evidently perpendicular to PQ, therefore so is MT. Now project, &c.

[N.B. This could be equally easily done by applying the result of Art. 292 at once to the ellipse, without using projections.]

7. Let a parallelogram be circumscribed to a circle, and let its corners be O_1, O_2, O_3, O_4. Let P and D be two adjacent points of contact with the circle, and let angle $DCP=\alpha$. Hence area of parallelogram $=4a^2 \operatorname{cosec} \alpha$. But the area of square circumscribing the circle is $4a^2$; hence $n=\operatorname{cosec} \alpha$.

Consequently $\sqrt{(n^2+n)} \pm \sqrt{(n^2-n)} = n\left\{\sqrt{\left(1+\frac{1}{n}\right)} \pm \sqrt{\left(1-\frac{1}{n}\right)}\right\}$

$$=\frac{1}{\sin \alpha}\left\{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2} \pm \left(\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}\right)\right\}$$

$$=\sec\frac{\alpha}{2} \text{ or } \operatorname{cosec}\frac{\alpha}{2}, \text{ according to sign.}$$

Also $$\frac{CO_1}{CP}=\sec\frac{\alpha}{2} \text{ and } \frac{CO_2}{CP}=\operatorname{cosec}\frac{\alpha}{2}.$$

Hence O_1, O_2, O_3, O_4 lie on two circles concentric with the given one, and with radii in the specified ratios; consequently all three circles will project into similar ellipses with the specified ratios between their axes.

8. Fully worked in the Answers.

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